

# Robustness of the Critical Behaviour in a Discrete Stochastic Reaction-Diffusion Medium\*

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**Abstract.** We study the steady states of a reaction-diffusion medium modelled by a stochastic 2D cellular automaton. We consider the Greenberg-Hastings model where noise and topological irregularities of the grid are taken into account. The decrease of the probability of excitation changes qualitatively the behaviour of the system from an “active” to an “extinct” steady state. Simulations show that this change occurs near a critical threshold; it is identified as a nonequilibrium phase transition which belongs to the directed percolation universality class. We test the robustness of the phenomenon by introducing persistent defects in the topology : directed percolation behaviour is conserved. Using experimental and analytical tools, we suggest that the critical threshold varies as the inverse of the average number of neighbours per cell.

**Keywords:** Reaction-diffusion media, stochastic cellular automata, phase transitions.

*Foreword:* See <http://webloria.loria.fr/~fates/Amybia/reacdiff.html> for accessing the complete set of experiments this paper refers to. The simulations presented in this paper were made with the *FiatLux* [1] cellular automata simulator.

## 1 Introduction

Let us consider the Greenberg-Hastings cellular automaton (GHCA) for the simulation of travelling-wave propagation in an excitable medium [2]. This model, and its different variants, have been intensively studied as a means of capturing the qualitative properties of various natural systems. It was also suggested that reaction-diffusion media could provide a path for building new types of computing devices [3]. For instance, we have recently proposed a bio-inspired system based on GHCA, to achieve decentralized and robust gathering of mobile agents scattered on a surface or computing tasks scattered on a massively-distributed computing medium [4]. As usual with such models, GHCA has mainly been studied using an homogeneous and regular lattice. However, in the context of massively distributed computing, one also needs to consider unreliable elements

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and defect-based noise. A first analysis showed that in this case, phase transitions could govern the behaviour of the system [4]. Our goal now is to broaden the knowledge on stochastic reaction-diffusion media by investigating how such systems behave when various types of noise are introduced.

Specifically, we focus on two types of noise or heterogeneities : (a) *temporary failures*: the excitation from one cellular automata cell may temporarily fail to propagate to a nearest-neighbour cell; (b) *definitive failures*: we consider two types of definitive failures; (1) a cell never gets excited; this is equivalent to adding “holes” in the lattice; (2) a cell permanently fails to have influence on one of its nearest-neighbours; this is equivalent to deleting links in the lattice. We show that variation of the probability to receive an excitation triggers a non-equilibrium phase transition between an “active” and an “extinct” macroscopic state and we study how the critical threshold varies when definitive failures are introduced in the model.

## 2 Model and Methods

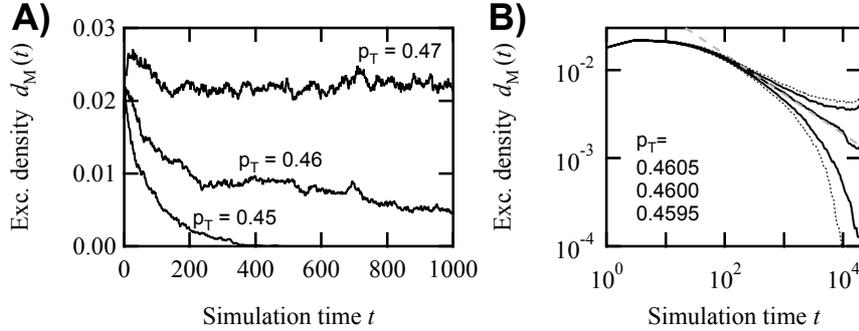
### 2.1 The Model

We study a two-dimensional stochastic version of the GHCA, where space is modelled by a square lattice  $\mathcal{L} = \{1, \dots, L\} \times \{1, \dots, L\}$  and each cell is denoted by its coordinates  $c = (c_x, c_y) \in \mathcal{L}$ . The set of possible states for each cell is  $\{0, \dots, M\}$ : state 0 is the *neutral* state, state  $M$  is the *excited* state and states 1 to  $M - 1$  are the *refractory states*. Each cell  $c$  is associated with a set of cells  $\mathcal{N}_c$ , its *neighbourhood*, which captures the locality of the influences between cells (see below). We denote by  $\sigma_c^t$  the state of a cell  $c$  at time  $t$ . Let  $E_c^t$  be the set of excited cells in  $\mathcal{N}_c$  at time  $t$ :  $E_c^t = \{c' \in \mathcal{N}_c \mid \sigma_{c'}^t = M\}$ . The cells of the GHCA are updated according to:

$$\sigma_c^{t+1} = \begin{cases} M & \text{if } \sigma_c^t = 0 \text{ and } \text{card}\{E_c^t\} > 0; \text{ with probability } p_T, \\ \sigma_c^t - 1 & \text{if } \sigma_c^t \in \{1, \dots, M\}, \\ 0 & \text{otherwise .} \end{cases} \quad (1)$$

The value of  $(1 - p_T)$  represents the rate of temporary excitation failures in the dynamics (i.e. noise).

In this paper, we explore the effects of different neighbourhoods on this system: (a) 4-nearest-neighbours (NESW adjacent cells); (b) 8-nearest-neighbours (adjacent + diagonal cells); (c) 6-nearest-neighbours (adjacent cells on a hexagonal grid) and (d) circular neighbourhoods [5]. We present results for periodic (toroidal) boundary conditions but we checked that bounded conditions (cells on the boundaries have less neighbours) give identical results. We use two methods to model definitive failures in the system: (1) We choose some cells at random before starting the simulation and definitely remove these cells from the lattice (i.e.  $\sigma_c^t = 0, \forall t$  for these cells). Missing cells, or *holes*, are selected with probability HR (hole rate). (2) We first construct the regular neighbourhood of each cell, then we remove each link to a neighbour with probability MLR (missing link rate).



**Fig. 1.** Evolution of the density of excited cells  $d_M(t)$ . (A) Single run with three values of the transmission rate: sub-critical  $p_T = 0.45$ , critical  $p_T = 0.46$ , supra-critical  $p_T = 0.47$ . (B) Log-log plot of the average of 30 runs for, from bottom to top (full lines) :  $p_T = 0.4595, 0.4600$  and  $0.4605$ . The dotted lines indicate  $-$  and  $+$  1 s.d. for  $p_T = 0.4595$  and  $0.4605$ , respectively. The gray dashed line indicates a decay compatible with the behavior expected for 2D directed percolation :  $d_M(t) \propto t^{-0.451}$ .  $L = 400$  (Left) or  $800$  (right); von Neumann topology and  $M = 4$ .

## 2.2 Effect of Noise

To evaluate the effects of local temporary failures (noise) on the *macroscopic* behaviour of the model, we monitored the evolution with simulation time of a macroscopic quantity, the overall density of excited cells  $d_M(t)$  (i.e. the total number of cells in state  $M$  at time  $t$  divided by the total number of cells). Fig. 1A shows prototypical evolutions of  $d_M(t)$  for different failure levels. When excitation probability  $p_T$  is high, the system reaches a steady-state where  $d_M$  oscillates around a fixed value. In this steady state, excitation waves fluctuate in space and time, but their overall number is rather stable. This state can thus be referred to as the *active* steady-state. Now, when temporary failures are too frequent (i.e., for low values of  $p_T$ ), the number of excited cells rapidly vanishes. The system reaches an *extinct* state, where all cells are neutral and where the activity is definitively washed out.

Plotted in log-log coordinates (Fig. 1B), the convergence to the active steady state corresponds to an upward curvature of the density while convergence to the extinct state is accompanied by a down-ward curvature (exponential decay). As shown in this figure, a very small change in the value of  $p_T$  has huge long-term effects, as it can decide which steady state, whether extinct or active, is reached. This indicates the presence of a phase transition around a *critical threshold*  $p_{TC}$ , that separates the  $p_T$  values leading to the active state from those leading to the extinct state. Statistical physics tells us that near the critical point, the laws governing the phase transition are power laws [6]. For instance in our case, one expects  $d_M(t) \sim t^{-\delta}$  at the critical threshold  $p_T = p_{TC}$  which means that, in log-log plots as in Fig. 1B, the critical threshold should manifest as a straight line. It is also a well-established fact that different systems may display identical critical exponent values (e.g.  $\delta$ ) [6], which means these systems share identical

fundamental properties. Systems sharing the same critical exponent values are said to belong to the same *universality class*.

Note that statistical physics predicts that  $\delta$  is a universal quantity (i.e., its value does not depend on details of the model) for which we have precise numerical estimations. By contrast, the location of the critical threshold  $p_{TC}$  is not universal and must be evaluated by other means. In the following, we used both simulations and mean-field techniques to study how this critical threshold depends on the lattice inhomogeneity. To test for the robustness of the phase transition, our method consisted in spotting changes of concavity in log-log curves and checking that the best-fit exponent thus obtained is compatible with the expected value of  $\delta$  (0.451 in our case, see [6,7,8]). Using this approach, we refined the measures on  $p_{TC}$  until the accuracy is of the order of  $10^{-4}$ . In most cases, it was sufficient to take a grid size of  $L = 800$ , an observation time  $T = 20000$  and to average the data over less than  $S = 40$  runs.

### 3 Robustness Studies

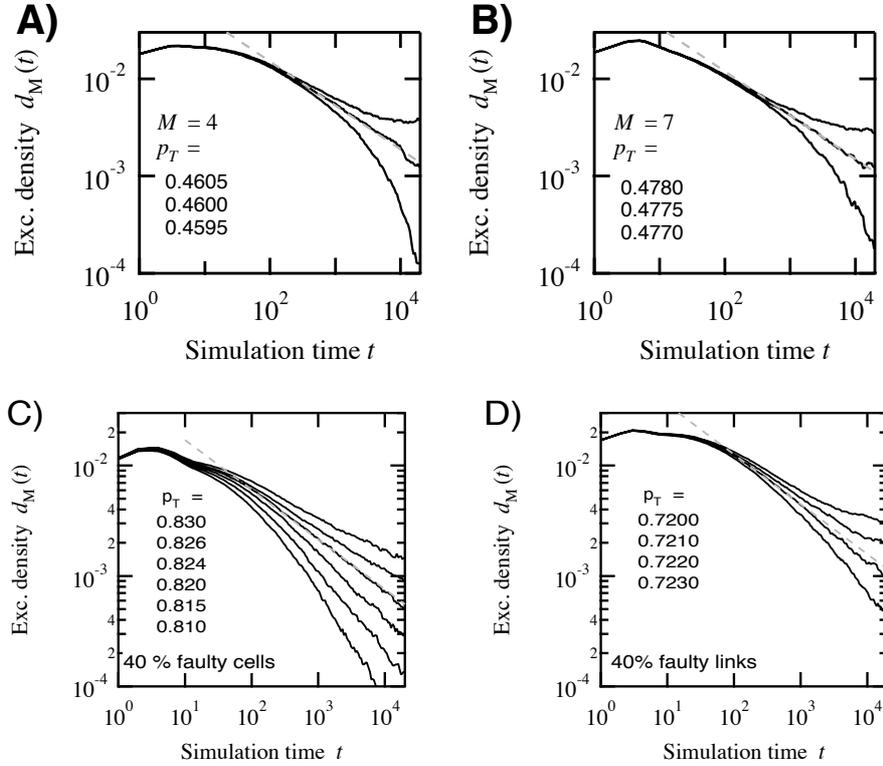
#### 3.1 Robustness to Variations of the Excitation Level

As a first step for the analysis of the robustness of the phase transition, we varied  $M$  and examined how  $p_{TC}$  is modified. Fig. 2A and B show the decay of the density of excited cells  $d_M(t)$  during simulations carried out with  $M = 4$  or 7. First of all, we observed that the critical exponent  $\delta$  was in both cases compatible with directed percolation (i.e., the values determined from the simulations were statistically compatible with those for directed percolation). Beyond the data presented in this figure, we evaluated  $\delta$  for  $M$  values ranging from 2 to 10 and various topologies (see our website<sup>1</sup>). We noticed that  $p_{TC}$  slightly increases with  $M$  (from 0.4245 to 0.4947 when  $M$  ranges from 2 to 10). At first sight, one may judge this variation moderate (less than 17%). It is an open question to determine if there is an asymptotic behaviour of the system for large values of  $M$ . From a practical point of view, an increase in  $M$  results in the apparition of longer waves, which implies that larger grids are required to measure the phase transition. However, for all the values of  $M$  we tested, the phase transition did belong to the universality class of directed percolation.

#### 3.2 Holes and Missing Links

To evaluate the effects of permanent failures on the phase transition, we varied the hole rate HR, and the missing link rate MLR, from 0 to 0.40 by increments of 0.10. As expected, the critical threshold increased with the defect rate. More surprisingly, we observed that as the defect rates increased, the critical threshold  $p_{TC}$  became more difficult to determine. The limits of our protocol were reached for HR = 0.40 or MLR = 0.40, see Fig. 2C and D. In both cases (faulty cells or links), the limitation stemmed from the difficulty to clearly spot the changes

<sup>1</sup> See <http://webloria.loria.fr/~fates/Amybia/reactdiff.html>



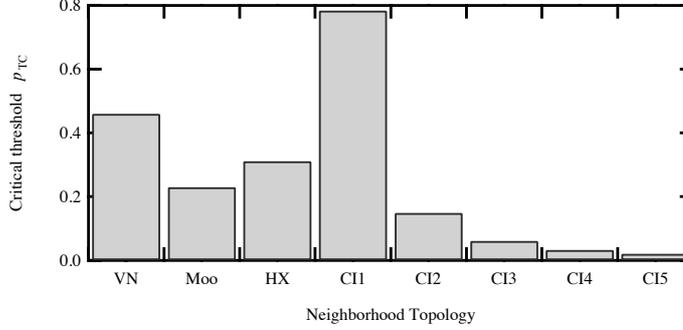
**Fig. 2.** (A-B) Robustness of the phase transition to the number of excitation level  $M$ . Decay of the instantaneous densities of excited cells  $d_M(t)$  for  $M = 4$  (A) and 7 (B). (C-D) Robustness of the phase transition with defects in the lattice. 40% of the cells (C) or of the links (D) were destroyed. The curves are obtained with  $S = 15$  runs, each run starts from different initial conditions and defect configurations. Simulation conditions :  $M = 4$ ,  $L = 1200$  ; von Neumann topology. Gray dashed lines as in fig.1.

of curvatures. We observed that even in large-size simulations, sub- and supra-critical curves are close to a power law. Experiments show that it is not possible to compensate this difficulty by a mere increase of the lattice size; it is an open problem to understand this “blurring effect” with more details.

However, it is impressive to observe that in all the experiments we conducted,  $\delta$  remained compatible with directed percolation (other simulations are available on our web site). Hence, albeit the precision is lower, these simulations show that the phase transition and its universality class are conserved in our model, even when close to half of the cells or links are destroyed.

### 3.3 Changing the Topology

Fig. 3 shows determination of critical threshold  $p_{TC}$  for various topologies and  $M = 4$ . Clearly, we observed that the critical threshold varies greatly with the



**Fig. 3.** Critical values of the phase transition for different topologies : VN : Von-Neuman; Moo : Moore; HX : Hexagonal; CIx : Circular with radius x. Error estimates on the values are less than  $10^{-3}$  in all cases, thus not visible on the figure.  $M = 4$ .

topology. To understand the origin of this variation, we measured the critical threshold and exponent for *more than 20 topologies*. These simulations showed that, again, directed percolation seems to remain the universality class of the observed phase transition *whatever the topology*. Interestingly enough, the data of Fig. 3 suggested that there exists an inverse proportionality law between the value of  $p_{TC}$  and the size of neighbourhood.

#### 4 The Inverse Proportionality Law

We now present an approximation of the behaviour of the model Eq.(1) using mean-field arguments. We only give here a rapid description of this analysis, leaving a more complete account of the calculus for future presentations. Let  $d_i(t)$  be the density of cells in state  $i$  at time  $t$ . Conservation principles derived from the transition rules can be expressed as a set of ordinary differential equations:

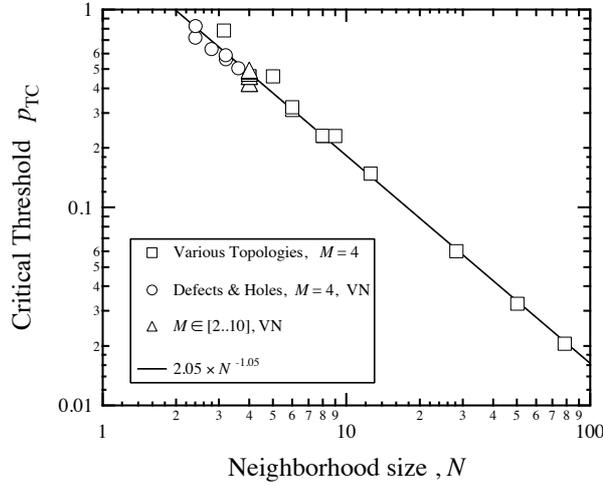
$$\begin{cases} \dot{d}_0(t) = d_1(t) - p_T d_0(t) \pi_0(t) \\ \dot{d}_j(t) = d_{j+1}(t) - d_j(t); \quad \forall j \in \{1, \dots, M-1\} \\ \dot{d}_M(t) = p_T d_0(t) \pi_0(t) - d_M(t) \end{cases} \quad (2)$$

where  $\pi_i(t)$  is the conditional probability that a cell  $c$  in state  $i$  has at least one excited cell in its neighbourhood. Under several mean-field-type assumptions (e.g., spatial correlation are neglected, cell states are assumed homogeneously distributed over space), we find that the steady-states for the density of excited cells  $d_M^*$  obey:

$$d_M^* - p_T (1 - M d_M^*) \left[ 1 - (1 - d_M^*)^N \right] = 0. \quad (3)$$

Eq. (3) has two solutions for a given  $p_T$ : one trivial steady-state  $d_{M,1}^* = 0 \forall p_T$  (“extinct” steady-state) and an active one,  $d_{M,2}^*$  that depends on  $p_T$ . Importantly, these two steady-states intersect for some unique value of  $p_T$ :  $d_{M,1}^* =$

$d_{M,2}^*$  for  $p_T = p_T^*$ . For  $p_T < p_T^*$ , the mean-field approach predicts that the system reaches the extinct state ( $d_{M,1}^* = 0$ ) asymptotically, while the active state ( $d_{M,2}^*$ ) is attained for  $p_T > p_T^*$ . This means  $p_T^*$  is the mean-field approximation for  $p_{TC}$ . To evaluate  $p_T^*$ , a series expansion at order 1 for  $d_M^* \rightarrow 0$  approximates Eq.(3) as  $(1 - p_T N) d_M^* + O(d_M^{*2}) \approx 0$ . Note that  $M$  has vanished from the order 1 term in this approximation.  $p_T^*$  is given by  $(1 - p_T^* N) = 0$ , so that one gets  $p_{TC} \approx 1/N$ . This indicates that  $p_{TC}$  should scale as the inverse of  $N$ . Moreover,  $p_{TC}$  is expected to depend on the average neighbourhood size  $N$  but not on the excitation level  $M$ .



**Fig. 4.** Illustration of the proportionality law between  $p_{TC}$  and  $1/N$ . Each data point reports the value of the critical threshold for a given condition as a function of the corresponding neighbourhood size  $N$  (the average number of neighbours). The red line shows the best fit to a power law  $p_{TC} = 2.05 \times N^{-1.05}$ .

This prediction is tested on Fig. 4. The critical thresholds for all the simulation we have run (various topologies, various defect rates, various  $M$  values) were plotted as a function of the corresponding neighbourhood size  $N$ . Strikingly, all the points collapsed on a single line in log-log coordinates. This confirms the power law dependence of  $p_{TC}$  on  $N$ . Fitting over all the data we obtained, we get:

$$p_{TC} \approx 2/N. \quad (4)$$

Therefore, our mean-field approach nicely predicts the observed  $N^{-1}$  dependency of the critical threshold. It thus shows that the important quantity to determine the value of the critical threshold is the average number of neighbours per cell and not the detail of the topology or its homogeneity. The mean-field approach however fails to predict several quantitative features of the simulations. First of all, the value of the prefactor in the scaling law Eq. 4 was predicted to be 1,

whereas the simulations rather indicate a value of 2. More intriguingly, the mean field approach predicts that  $p_{TC}$  does not depend on the number of cell states  $M$ . Yet, we have already seen above that this is not observed in our simulations:  $p_{TC}$  increases slightly with  $M$ . However, this increase is moderate enough that the observed values are still roughly in agreement with the  $N^{-1}$  scaling, as seen in Fig 4 (triangles).

## 5 Conclusion

The observations reported in this paper confirmed the great robustness of the directed percolation universality class. Although this statement is not new, to our knowledge, it is the first time that this robustness was confronted against such a large number of perturbations: holes in the grid, missing links, change of topology. This impressive robustness makes the phase transition in this model an interesting phenomenon for the design of novel computing devices.

Note that a *conjecture* by Janssen and Grassberger (see [6] for an overview of this topic) lists the fundamental properties that a system must display in order to belong to this universality class. Clearly, in spite of the added noise and heterogeneity, the different systems we considered all validate these conditions, so that the amazing robustness of the phase transition in these systems unambiguously pleads in favour of the validity of this conjecture. However, as far as we know, there exists no theory to predict the value of the critical threshold. In a context of building computing devices that would use reaction-diffusion waves, determining the value of the critical threshold would be of paramount importance. The inverse proportionality law we presented paves the way for obtaining generic laws (even approximate ones) to predict the position of the critical threshold in various simulation conditions.

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