Simulation of fibrous material
Mechanical models and numerical simulation

Florence Bertails-Descoubes
BiPop - INRIA/LJK

Nonsmooth Mechanics Spring School, June 2010
Research area

Realistic and efficient simulation of macroscopic mechanical systems

**Context**

- **Visually relevant** phenomena
  Large scale, deformations, contacts

- **Increasing demand** in such simulators
  Computer graphics, cosmetology, medical area, ...

- **Pluridisciplinary** field
  Mechanics, numerical simulation, graphics

**Main research direction**

→ Search for **compact** models:
  Realism + robustness + efficiency
Fibrous materials: Motivation

• 1D structures, « strands », are wide spread
  – Hair, plants, trees, wires, etc.

• Typical features
  – Thickness << length
  – Inextensible
  – Bending and twisting elasticity
  – Large motion/applied forces
Example: Hair Simulation
**Problems**

- **Complex** mechanical system (nonlinear, lots of contacts, friction...)
- Multi-scale dynamics
- No reference mechanical model

Example: Hair Simulation
Motivation

*Increasing demand by industrial applications*

- Virtual entertainment (features films, special effects, games)
  

- Virtual prototyping in cosmetology

  © L’Oréal
General outline

I. Individual fiber: high-order rod model

II. Assembly of fibers with frictional contact (ongoing work)
Part I: Outline

I. Maximal vs. reduced coordinates models

II. Super-helix model (2006)

III. Linear-time super-helix model (2008)

IV. Conclusions
Part I: Outline

I. Maximal vs. reduced coordinates models

II. Super-helix model (2006)

III. Linear-time super-helix model (2008)

IV. Conclusions
Models for elastic rods: 2 families

1. Maximal coordinates

   - # parameters > # dof (ex: 3D points)
   - Constraints for the kinematics
     (inextensibility + bending + twisting)
   - **Examples:**
     - Spline dynamics [Lenoir et al. 2004, Thetteen et al. 2008]
     - Discrete Cosserat [Spillmann et al. 2007, Bergou et al. 2008]
   - **Pros & cons:**
     - 😊 Generally, sparse system to solve
     - 😊 Simplifies the handling of interactions
     - 😞 Constraints difficult to formulate (twist, inextensibility)
     - 😞 Possible drifts
     - 😞 Redundancy of parameters
2. Generalized (reduced) coordinates

- # parameters = # dofs (≠ 3D points)
- No extra constraint

Examples:
- Serial chain of rigids [Hadap et al. 2001, Hadap 2006]
- Super-Helices [Bertails et al. 2006]

Pros & cons:
- 😊 Intuitive and minimal parameterization
- 😊 Perfect inextensibility $\rightarrow$ « nervous » motions possible
- 😞 Contact “tricks” used in graphics not adaptable
- 😞 Often, dense system $\rightarrow$ quadratic complexity

Models for elastic rods: 2 families
Part I: Outline

I. Maximal vs. reduced coordinates models

II. Super-helix model (2006)

III. Linear-time super-helix model (2008)

IV. Conclusions
Hair strand = elastic rod

- very small thickness
- elastic
- inextensible
Cosserat rod

- Well-known in mechanics
- Shape given by
  - centerline $r(s)$
  - material frame $n_0(s), n_1(s), n_2(s)$
- Kirchhoff assumptions
  - no shearing
  - inextensible
Kirchhoff rod

- 3 modes of deformation
  - 1 mode of twist
  - 2 modes of bending

- Parameterization of an inextensible rod by
  - the twist $\tau(s)$
  - the curvatures $\kappa_1(s)$ and $\kappa_2(s)$

- $\Omega$ is the rate of rotation

\[ \Omega(s) = \tau(s) n_0(s) + \kappa_1(s) n_1(s) + \kappa_2(s) n_2(s) \]
Dynamics of Kirchhoff’s rods

- **Kirchhoff’s equations**
  \[
  \rho S \frac{\partial^2 \kappa}{\partial t^2} + EI \frac{\partial^4 \kappa}{\partial s^4} + \text{nonlinear terms} = 0
  \]

- **Numerical integration is difficult**
  - stiff problem (inextensibility condition)
  - we tried best schemes from mech. eng.
  - not suited for hair anim. \[\text{[Hou&Klapper1998]}\]

- **Super-Helix model**
  - new deformable model
  - animated using Lagrangian mechanics
Key idea

- Classical nodal schemes
  - Approximate spatial derivatives
  - Ok if the rod is extensible [CORDE, Spillman 2007]
  - Unstable if the rod is inextensible (stiff system)

- If we look at a rod with constant \((\tau(s), \kappa_1(s), \kappa_2(s))\)
  - Its shape is exactly a circular helix (\(\Omega\) constant)
  - Kinematics terms can be computed analytically

\[
\frac{\partial \kappa}{\partial s} \approx \frac{\kappa_{i+1} - \kappa_i}{\Delta s}
\]

\[
r(s) = f(s, \tau, \kappa_1, \kappa_2)
\]

\[
\dot{r} = f(s, \tau, \kappa_1, \kappa_2, \kappa_1, \dot{\tau}, \dot{\kappa}_2)
\]
Kinematics of a Super-Helix

- Split rod into $N$ elements
- Take $(\tau(s), \kappa_1(s), \kappa_2(s))$ piecewise $C^0$
- $r(s)$ is $C^1$-smooth at elements boundaries
- Generalized coordinates $q$ defined by

$$q = \{\tau^1, \kappa_1^1, \kappa_2^1, \ldots, \tau^N, \kappa_1^N, \kappa_2^N\} \in \mathbb{R}^{3N}$$
A single element (generic helix shape) can be:
• a circle
• a line, twisted or not
• a helix with several spires
Shapes of Super-Helices

Thus a wide range of hair types …

smooth → wavy → curly → fuzzy
Example of kinematics

Typical configurations: tuning $q$ for $N = 5$ elements
We have defined the shapes

• Kinematics computed analytically
  → no error approximation in space

• Let’s turn to the dynamics!
Dynamics

- Use an old cookbook recipe by Lagrange
  \[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial (T - U)}{\partial q} = F \]
- Add Kirchhoff sauce
  \[ T(q, \dot{q}) = \frac{1}{2} \int_0^L \rho S \hat{r}^2 ds \quad \text{and Kirchhoff’s} \ U(q) \]
- Resulting equations of motion read
  \[ M(q) \cdot \ddot{q} + K \cdot q = I(q, \dot{q}) + F \]
Super-Helix simulations

No damping
Super-Helix simulations

Damping by rod’s viscosity
Super-Helix simulations

Damping by fluid's viscosity
Super-Helix simulations

Various natural curvatures and twist
Qualitative validation

Estimating fibers parameters
- radius
- ellipticity
- curliness
- density
- stiffness

Calibrating friction
- visco-elasticity $\gamma$
- air viscous drag $\nu$

Measured
Reported values

Super-Helix

Input

Input

250 $\mu$m

$2r_h$

$\Delta_h$
Spring experiment

Real hair wisp

Simulation
Twist experiment

Twisting
Plectoneme experiment

See movie!
Advantages

- Accurate modeling of **bending** and **twisting** effects
- Perfect **inextensibility** (kinematic constraint)
- **Various** rest shapes
- High-order elements → **rich** set of representations
- **Intuitive** and **minimal** parameterization
- Some experimental **validation**
Major drawback

- **Quadratic** complexity in the number of elements

N = 5 elements  
160 frames/sec

N = 20 elements  
4 frames/sec

→ High *precision* ruins *performance*
Part I: Outline

I. Maximal vs. reduced coordinates models

II. Super-helix model (2006)

III. Linear-time super-helix model (2008)

IV. Conclusions
Recursive Super-Helices

- Chains of articulated rigid bodies
  → Problem has been solved, by [Featherstone1983] (among others)

- Super-Helices
  → A new challenge to be solved

Idea: **recursive** formulation (à la Featherstone)

Advantages:
- **Linear** complexity
- Straightforward modeling of **tree-like** structures
Featherstone’s algorithm (1983)

- Serial chain of rigid bodies
  - N+1 rigid bodies, N joints
  - N parameters $q_Q$ (generalized coordinates)

- Featherstone’s Articulated-Body-Algorithm
  - Exploits the following affine relationship:
    $$ a_Q = H_Q \ddot{q}_Q + b_Q $$
    spatial acceleration of rigid Q
  - Formulate the dynamics locally:
    $$ f_Q = I_A^Q a_Q + p_A^Q $$
    force transmitted onto rigid Q through joint Q

where $I_A^Q$ and $p_A^Q$ are recursively computed from the free end (indep. of $\dot{q}_i$)
Super-Helices vs Serial chains

Super-Helix $\neq$ Serial chain of rigid bodies:

- Deformable elements $\neq$ rigid bodies
- Finite elements model $\neq$ articulated system
However…

Significant common features:

- **Recursive** kinematics
- **Affine** relationship between 3d acc. and generalized acc.
Recursive kinematics

- **Acceleration of element** $Q$
  \[
  \ddot{\mathbf{r}}_Q(u) = \ddot{\mathbf{r}}_Q(0) + A^r_Q(u) \dddot{k}_Q + C^r_Q(u) \dddot{\theta}_Q(0) + b^r_Q(u)
  \]
  \[
  \ddot{\theta}_Q(u) = \ddot{\theta}_Q(0) + A^\theta_Q(u) \dddot{k}_Q + b^\theta_Q(u)
  \]

- **Smoothness between elements**
  \[
  \begin{align*}
  \ddot{\mathbf{r}}_Q(0) &= \ddot{\mathbf{r}}_{Q-1}(\ell_{Q-1}) \\
  \ddot{\theta}_Q(0) &= \ddot{\theta}_{Q-1}(\ell_{Q-1})
  \end{align*}
  \]

- **Conditions at clamped end** ($Q = 1$)
  \[
  \ddot{\mathbf{r}}_1(0) = \ddot{\mathbf{r}}_{cl} \quad \ddot{\theta}_1(0) = \ddot{\theta}_{cl}
  \]
Recursive kinematics

- Kinematics is very close to that of serial chains
- What about dynamics?
Recursive dynamics?

- Global equation for a Super-Helix

\[
M(t, \kappa) \ddot{\kappa} + K(\kappa - \kappa^0) = B(t, \kappa, \dot{\kappa}) \quad 3N \text{ scalar equations}
\]

- Local equation on element \( Q \)?
  
  - Local term: \( K(\kappa - \kappa^0) \rightarrow K_Q(\kappa_Q - \kappa_Q^0) \quad \forall Q \in \{1..N\} \)
  
  - Global terms: \( M(t, \kappa) \ddot{\kappa} B(t, \kappa, \dot{\kappa}) \)

\[
M(t, \kappa) \ddot{\kappa} + K(\kappa - \kappa^0) = B(t, \kappa, \dot{\kappa}) - M(t, \kappa) \ddot{\kappa} \\
K_Q(\kappa_Q - \kappa_Q^0) = G_Q \quad \forall Q \in \{1..N\}
\]

Problem: depends on all \( \ddot{\kappa}_i, i \in \{1..N\} \)
Key of the method

One can prove that:

• $G_Q$ can be expressed **locally** on element $Q$

\[ G_Q = A_Q^A \ddot{\kappa}_Q + R_Q^A \dot{r}_Q(0) + C_Q^A \dot{\theta}_Q(0) + b_Q^A \]

• Where $A_Q^A$, $R_Q^A$, $C_Q^A$, $b_Q^A$ are **recursively** computed from $Q + 1$ to $Q$

• With the BC at free end **known**

(zero transmitted force)
Final recursive algorithm

Two main passes:

1. From the free end to the clamped end
   - Initialization at the free end, at $T+1$
   - Recursive computation of the $A_Q^A$, $R_Q^A$, $C_Q^A$, $b_Q^A$

2. From the clamped end to the free end
   - Initialization at the clamped end, at $T+1$
   - For current element $Q$, solve the local equation

\[
K_Q (\kappa_Q - \kappa_Q^0) = A_Q^A \dddot{\kappa}_Q + R_Q^A \dddot{r}_Q(0) + C_Q^A \dddot{\theta}_Q(0) + b_Q^A
\]

$\rightarrow \kappa_Q, \dddot{\kappa}_Q$ at $T+1$

- Kinematics of $Q \rightarrow$ Initialization of $Q + 1$
Extension to tree-like structures

Two main passes:

1. From the free ends to the clamped end
   - Initialization at the free ends, at \( T+1 \)
   - Recursive computation of the \( A_Q^A, R_Q^A, C_Q^A, b_Q^A \)

2. From the clamped end to the free ends
   - Initialization at the clamped end, at \( T+1 \)
   - For current element \( Q \), solve the local equation
     \[
     K_Q (\kappa_Q - \kappa_Q^0) = A_Q^A \ddot{\kappa}_Q + R_Q^A \dot{r}_Q(0) + C_Q^A \ddot{\theta}_Q(0) + b_Q^A
     \]
   - Kinematics of \( Q \) \( \rightarrow \) Initialization of the descendants of \( Q \)
Results

Linear-time Super-Helices

Florence Bertails

August 2009
## Performance results

<table>
<thead>
<tr>
<th>Model (N elements)</th>
<th>FPS recursive</th>
<th>FPS composite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight rod (1)</td>
<td>170</td>
<td>170</td>
</tr>
<tr>
<td>Straight rod (2)</td>
<td>170</td>
<td>170</td>
</tr>
<tr>
<td>Straight rod (10)</td>
<td>120</td>
<td>40</td>
</tr>
<tr>
<td>Straight rod (20)</td>
<td>54</td>
<td>4</td>
</tr>
<tr>
<td>Straight rod (30)</td>
<td>37</td>
<td>1.4</td>
</tr>
<tr>
<td>Palm tree (8)</td>
<td>170</td>
<td>-</td>
</tr>
<tr>
<td>Algae (10)</td>
<td>108</td>
<td>-</td>
</tr>
<tr>
<td>Complex 3D tree (50)</td>
<td>25</td>
<td>-</td>
</tr>
<tr>
<td>Super-Helix Man (81)</td>
<td>17</td>
<td>-</td>
</tr>
<tr>
<td>Weeping willow (242)</td>
<td>6</td>
<td>-</td>
</tr>
</tbody>
</table>
Performance results

N = 5 elements

- Composite: 160 frames/sec
- Recursive: 160 frames/sec

N = 20 elements

- Composite: 4 frames/sec
- Recursive: 54 frames/sec
Stability issue

• Integration « less implicit » compared to composite method
  → dt has to be smaller for a stable simulation

• However:
  – small difference
  – does not compensate the gain in linear complexity
Part I: Outline

I. Maximal vs. reduced coordinates models

II. Super-helix model (2006)

III. Linear-time super-helix model (2008)

IV. Conclusions
Conclusions

Summary
• Super-helix: a new high-order model for inextensible elastic rods
• Linear-time dynamics of super-helices made possible
  → Complex configurations are efficiently simulated
  → Natural extension to tree-like structures of super-helices

Future work
• Inverse statics for initializing parameters from input geometry
• Adaptive simulation of rods (similar to [Redon et al. 2005])
• Simulation of contact/friction (to be continued in part II)
• ...