Four Lectures on Differential Variational and Complementarity Systems with Applications

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reporting joint work with co-authors
Lecture I:
Non-Zenoness of a Class of Differential Quasi-Variational Inequalities

Lecture II:
Continuous-time Dynamic User Equilibrium

Lecture III:
Linear-Quadratic Optimal Control with Mixed State-Control Constraints

Lecture IV:
Differential Linear-Quadratic Nash Games with Mixed State-Control Constraints
Lecture I

Non-Zenoness of a Class of Differential Quasi-Variational Inequalities

Monday September 10, 2012, 10 AM – 11 AM
Contents of Presentation

• Definition of the differential quasi-variational inequality (DQVI)

• A frictional contact problem with local compliance

• Mathematical programming background

• Existence of a solution trajectory

• System modes and (non-)Zeno states

• The main non-Zenoness theorem

• A sketch of the inductive proof

• An expansion lemma via Lie derivatives
Some references


L. Han, K. Camlibel, J.S. Pang, and W.P.M.H. Heemels. A unified numerical scheme for linear-quadratic optimal control problems with joint control and state constraints. *Optimization Methods and Software*, in print
The Mathematical Programming World thus far

Mathematical programming (MP) to date has been a **static** subject of study, yet is very effective for handling **constraints**, particularly inequalities.

<table>
<thead>
<tr>
<th>The progression in finite dimensions</th>
<th>minimize $c^T x + \frac{1}{2} x^T Q x$</th>
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<tbody>
<tr>
<td><strong>linear/convex quadratic program</strong></td>
<td>$x \in P \subseteq \mathbb{R}^n$</td>
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<td><strong>linear complementarity problem (LCP)</strong></td>
<td>$0 \leq z \perp q + Mz \geq 0, ; z \in \mathbb{R}^n$</td>
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<td><strong>nonlinear complementarity problem (NCP)</strong></td>
<td>$0 \leq z \perp F(z) \geq 0$</td>
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<tr>
<td><strong>variational inequality (VI)</strong></td>
<td>$z \in K: (y - z)^T F(z) \geq 0$</td>
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<td><strong>SOL(K, F)</strong></td>
<td>for all $y \in K \subseteq \mathbb{R}^n$</td>
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<td><strong>many extensions.</strong></td>
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The Differential World thus far

- Classical dynamical systems are **unconstrained** and its applications are largely restricted to systems with **smooth** transitions.

<table>
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<th>From the</th>
<th>$\dot{x} = f(x), \ x \in \mathbb{R}^n$</th>
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<tr>
<td>ordinary differential equation (ODE)</td>
<td>$\dot{x} = F(x, y)$</td>
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<td>$0 = G(x, y)$</td>
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<td>differential algebraic equation (DAE)</td>
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<tr>
<td>differential inclusion (DI)</td>
<td>$\dot{x} \in \Phi(x) \subseteq \mathbb{R}^n$</td>
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- Time evolutionary systems are dynamic in nature, yet exhibit mode switches, cannot be modeled by MP and ODEs separately,

- Need a framework to capture: **dynamics and mode switch**.

- **Birth of the Differential Variational/Complementarity system:**
  - a novel framework that occupies a unique niche between a DAE and a DI
  - extending the DAE to include inequalities
  - specializing the DI to a multi-valued $\Phi$ being the solution map of a VI.
Differential variational inequalities (DVIs)

form a novel mathematical paradigm that offers a broad, unifying framework for modeling many applied (dis)equilibrium problems containing

• dynamics (pathway to equilibrium)
• inequalities (unilateral constraints), and
• disjunctions (transitions).

The Differential Complementarity System (DCS)

Given $(F, G, H) : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m+\ell}$, and $x^0 \in \mathbb{R}^n$, find continuous-time trajectories $(x, y, z) : [0, T] \rightarrow \mathbb{R}^{n+m+\ell}$ such that

\[
\begin{align*}
\dot{x} &= F(x, y, z), \quad x(0) = x^0 \quad \text{dynamics} \\
0 &= G(x, y, z) \quad \text{algebraic equations} \\
0 \leq z \perp H(x, y, z) &\geq 0 \quad \text{inequalities and disjunction}
\end{align*}
\]

where $\perp$ is the perpendicularity notation, which in this context denotes the well-known complementary slackness condition in optimization.
The Differential Quasi-Variational Inequality

Find time-dependent trajectories \( x(t), y(t), \) and \( u(t) \) such that

\[
\dot{x} = A(x, y) + B(x, y)u \quad \text{an ordinary differential equation}
\]

\[
0 \leq y \perp G(x, y) \geq 0 \quad \text{complementarity condition}
\]

\[
u \in \text{SOL}(K(x, y), C(x, y), N(x)) \quad \text{variational inequality},
\]

where

\[
\dot{x} \equiv dx/dt \quad \text{the time-derivative of } x;
\]

\[
A : \mathbb{R}^{n+m} \to \mathbb{R}^n ; \quad B : \mathbb{R}^{n+m} \to \mathbb{R}^{n \times \ell} ; \quad G : \mathbb{R}^{n+m} \to \mathbb{R}^m ; \quad C : \mathbb{R}^{n+m} \to \mathbb{R}^\ell \quad N : \mathbb{R}^n \to \mathbb{R}^{\ell \times \ell} ; \quad H : \mathbb{R}^{n+m} \to \mathbb{R}^k ; \quad E \in \mathbb{R}^{k \times \ell} \quad \text{a constant matrix};
\]

\( \perp \) is the orthogonality notation, i.e., \( a \perp b \) means \( a^T b = 0 \); and

\[
K(x, y) \equiv \{ u \in \mathbb{R}^\ell : H(x, y) + Eu \geq 0 \},
\]

is a moving polyhedron undergoing translation, whose dependence on \( (x, y) \) is the quasi-nature of the system.
Solution set of the affine variational inequality (AVI) defined by the triple 
\((K(x,y), C(x,y), N(x))\), that is,

- \( u \in \text{SOL}(K(x,y), C(x,y), N(x)) \) if and only if \( u \in K(x,y) \) and
  \[
  (u' - u)^T(C(x,y) + N(x)u) \geq 0, \quad \forall u' \in K(x,y).
  \]
or equivalently,

- \( u \in \text{SOL}(K(x,y), C(x,y), N(x)) \) if and only if \( \exists \) a multiplier \( \lambda \) such that
  \[
  0 = C(x,y) + N(x)u - E^T\lambda \\
  0 \leq \lambda \perp H(x,y) + Eu \geq 0.
  \]

- \( \exists \gamma > 0 \) such that for all \((x,y)\) and for all \( u \in \text{SOL}(K(x,y), C(x,y), N(x)) \), a multiplier \( \lambda > 0 \) exists satisfying
  \[
  \| \lambda \| \leq \gamma \| C(x,y) + N(x)u \|.
  \]
As a Differential Complementarity System

\[ \dot{x} = A(x, y) + B(x, y)u \]

\[ 0 \leq y \perp G(x, y) \geq 0 \]

\[ 0 = C(x, y) + N(x)u - E^T\lambda \]

\[ 0 \leq \lambda \perp H(x, y) + Eu \geq 0 \]

- The polyhedrality of \( K(x, y) \) facilitates the application of AVI theory.
- The constancy of \( E \) could possibly be relaxed at the expense of significantly complicating the analysis.
- The above form already provides a broad framework with many interesting special cases.
- The complementarity conditions identify this as a nonsmooth, and possibly multi-valued, dynamical system.
Frictional contact with local compliance

Multiple bodies come into contact under external forces (gravitation e.g.); goal is to understand body motion (possibly deformations) and interaction of forces, some internal.

Friction at contact points induces rolling and slipping, leading to system mode changes, among other things. Also contacts can break.

Initiated by Löstedt (1981), rigid-body paradigm have been extensively examined; yet they induce discontinuity in velocities and impulsive forces, thus are challenging to analyze rigorously and simulate numerically.

Mathematically, rigid-body models require the theory and methods of measure differential inclusions.

Locally compliant models were proposed in Song’s 2002 Ph.D. thesis as an alternative to ease some deficiencies associated with the rigid-body models, without sacrificing the physics of contact.

It turns out these locally compliant models have nice mathematical properties.
Friction model with compliance:

5 main components

- Newton's law to describe force-induced body dynamics
- Kinematics to specify body orientation due to rotational motion
- Constitutive law of compliance
- Principle of non-penetration
- Friction law
Model Notations

Dimensions and constants

- $n_q$, $n_\nu$, $n_\delta$: positive dimensional integers
  (for generalized coordinates, velocities, and contacts, respectively)
- $\mu > 0$: $n_\delta$-dimensional vector of friction coefficients at the contacts
- $q^0$, $\nu^0$, $\delta^0$: initial states (generalized coordinates, velocities, and deformations, respectively) of the system;
- $T$: positive time duration.

Solution trajectories (all are functions of the time $t \in [0, T]$)

- $q$: $n_q$-vector of generalized coordinates of the bodies in contact
- $\nu$: $n_\nu$-vector of generalized velocities of the bodies
- $\lambda_{n,t,o}$: 3 $n_\delta$-vectors of contact forces in the normal, labeled “n”, and the two tangential directions, labeled “t” and “o”, respectively
- $\delta_{n,t,o}$: 3 $n_\delta$-vectors of the body deformations in the normal (n) and the two tangential directions
- $s_{n,t,o}$: 3 $n_\delta$-vectors of separations in the normal and two tangential directions
Model Notations (cont.)

Model functions (note their respective arguments)

\[ M(q) \quad n_\nu \times n_\nu \text{ symmetric positive definite mass-inertia matrix;} \]
\[ f(q, \nu) \quad n_\nu\text{-dimensional vector of external forces applied to the system} \]
\[ G(q) \quad n_q \times n_\nu\text{-parametrization matrix describing system orientation} \]
\[ G(q)^T G(q) \text{ equals the identity matrix of order } n_\nu \]
\[ \Psi_{n,t,o}(q) \quad 3 \times n_\delta\text{-dimensional distance and displacement functions in the normal and the two tangential directions} \]
\[ K(q) \quad 3n_\delta \times 3n_\delta \text{ symmetric positive definite system stiffness matrix} \]
\[ C(q) \quad 3n_\delta \times 3n_\delta \text{ symmetric positive (semi)definite system damping matrix} \]
The model equations

- Force equilibrium: \( M(q) \dot{\nu} = f(q, \nu) + W(q) \lambda \), where \( M(q) \) is symmetric positive definite;

- Kinematics: \( \dot{q} = G(q) \nu \);

- Definition of separation: \( s \equiv \delta + \Psi(q) \);

- Constitutive law of local compliance: \( \lambda = K(q) \delta + C(q) \dot{\delta} \), where \( K(q) \) and \( C(q) \) are both symmetric and positive (semi) definite;

- Normal non-penetration and contact: \( 0 \leq \lambda_n \perp s_n \geq 0 \);

- Tangential friction principle: for each contact point \( i \),

\[
(\lambda_{it}, \lambda_{io}) \in \arg \min_{(\hat{\lambda}_{it}, \hat{\lambda}_{io}) \in F_i(\mu_i \lambda_{in})} \left\{ \frac{d s_{it}}{dt} \hat{\lambda}_{it} + \frac{d s_{it}}{dt} \hat{\lambda}_{io} \right\},
\]
Friction cones

described by the cone-valued set-valued maps $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}^3$; prominent examples are:

The Lorentz cone: $\mathcal{F}^C(\rho) \equiv \{(a, b) \in \mathbb{R}^2 : \sqrt{a^2 + b^2} \leq \rho\}, \rho \geq 0$;

its polygonal approximations:
$\hat{\mathcal{F}}(\rho) \equiv \{(a, b) \in \mathbb{R}^2 : \cos \theta_j + b \sin \theta_j \leq \rho, \ j = 1, \ldots, \ell\}$
for various angles $\theta_j \in [0^\circ, 360^\circ)$ such that the polygonal region $\hat{\mathcal{F}}(1)$ either inscribes or circumscribes the unit circle;

a special cone: $\mathcal{F}^P(\rho) \equiv \{(a, b) \in \mathbb{R}^2 : \max(|a|, |b|) \leq \rho\}$,

 corresponding to $\ell = 4$ and $\theta_j \in \{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$. 
The case of no normal damping

Assume the damping matrix to be of the form:

\[
C(q) \equiv \begin{bmatrix}
0 & 0 & 0 \\
0 & C_{tt}(q) & C_{to}(q) \\
0 & C_{ot}(q) & C_{oo}(q)
\end{bmatrix}.
\]

Using a \textit{semismooth} implicit-function theorem, we obtain, with \(z \triangleq (q, \nu, \delta_t, \delta_o)\),

- \(\dot{z} = \Gamma(z)\), where \(\Gamma\) is a \textit{semismooth} function
- existence and uniqueness of a \(C^1\) solution trajectory \(z(\bullet, x^0)\) for every given \(z(0) = z^0\)
- \(z(t, \bullet)\) is a \textit{semismooth} function of initial condition
- a non-Zeno result under polygonal friction; implying that contacts of various kinds are short-time persistent
- finite (as opposed to impulsive) contact forces.
AVI Results

Notation. Let $\mathcal{P}_E(b) \equiv \{ u : b + Eu \geq 0 \}$ and $\mathcal{D}\mathcal{P}_E \equiv \{ b : \mathcal{P}_E(b) \neq \emptyset \}$.

For a positive definite matrix $N^*$ and vector $b \in \mathcal{D}\mathcal{P}_E$, let $\Phi_{(N^*,E)}(b,q)$ denote the unique solution of the AVI $(\mathcal{P}_E(b),q,N^*)$; i.e., $\Phi_{(N^*,E)}(b,q)$ is the unique vector $v^* \in \mathcal{P}_E(b)$ such that

$$(v - v^*)^T(q + N^*v^*) \geq 0, \quad \forall v \in \mathcal{P}_E(b).$$

Let $\text{pos}(E^T)$ denotes the polyhedral cone generated by the columns of $E^T$.

- $\Phi_{(N^*,E)}$ is a piecewise linear, thus Lipschitz continuous, function on its domain $\mathcal{D}\mathcal{P}_E \times \mathbb{R}^\ell$.

- $\Phi_{(N^*,E)}(0,q) = 0$ for all $q \in \text{pos}(E^T)$.

A perturbation lemma. Let $c : \mathbb{R} \rightarrow \mathbb{R}^\ell$ be a $(\nu - 1)$-order vector polynomial such that $c(t) \in \text{pos}(E^T)$ for all $t \geq 0$ sufficiently small. For every pair of vectors $(b,a) \in \mathcal{D}\mathcal{P}_E \times \mathbb{R}^\ell$, a vector $\xi$ exists such that $\Phi_{(N^*,E)}(t^\nu b, c(t) + t^\nu a) = t^\nu \xi$ for all $t \geq 0$ sufficiently small.  \[ \square \]
The Differential Quasi-Variational Inequality, restated

Find time-dependent trajectories $x(t)$, $y(t)$, and $u(t)$ such that

| $\dot{x}$ | $= A(x, y) + B(x, y)u$ | an ordinary differential equation |
| $0 \leq y \perp G(x, y) \geq 0$ | complementarity condition |
| $u \in \text{SOL}(K(x, y), C(x, y), N(x))$ | variational inequality, |
**Assumptions:** Given a pair \((x^0, y^0)\) satisfying the complementarity condition: 
\[0 \leq y^0 \perp G(x^0, y^0) \geq 0:\]

(A) the functions \(A, B, C, G, H,\) and \(N\) are analytic in a neighborhood of \((x^0, y^0);\)
[C\(^1\) suffices for well-posedness, analyticity is needed for non-Zenoness.]

(B) \(y^0\) is a strongly regular solution of the NCP: 
\[0 \leq y \perp G(x^0, y) \geq 0;\]

(C) the matrix \(N(x^0)\) is positive definite (albeit not necessarily symmetric);

(D) \(H(x, y) \in \mathcal{DP}_E\) for all \((x, y)\) in a neighborhood of \((x^0, y^0)\) with \(y \geq 0.\)

- For all \((x, y)\) near \((x^0, y^0)\) with \(y \geq 0,\) \(\text{SOL}(K(x, y), C(x, y), N(x))\) is a singleton whose unique element is \(u(x, y) \equiv \Phi_{(N(x), E)}(H(x, y), C(x, y)).\)

- The solution function \(u(x, y)\) for \(y \geq 0\) is well defined, Lipschitz continuous, and piecewise analytic in a neighborhood of \((x^0, y^0).\)
[A nontrivial result due to the translational nature of set \(K(x, y)\) and the variable matrix \(N(x).\)]
Strong Regularity (Robinson 1980)

• If $y^0$ is a strongly regular solution of the nonlinear complementarity problem (NCP): $0 \leq y \perp G(x^0, y) \geq 0$, then $\exists$ open neighborhoods $U$ and $V$ of $x^0$ and $y^0$, respectively, and a Lipschitz continuous function $y : U \to V$ such that for each $x \in U$, $y(x)$ is the unique vector in $V$ that solves the NCP at $x$.

• Moreover, if $G$ is analytic near $(x^0, y^0)$, then $y(x)$ is piecewise analytic near $x^0$.

• Strong regularity is preserved under pivoting (a standard operation in linear programming).

• There are matrix-theoretic characterizations of strong regularity in terms of the principal submatrices of the Jacobian matrix $J_yG(x^0, y^0)$. 
Existence of a local solution trajectory

In terms of the implicit solution functions $y(x)$ and $u(x, y)$,

$$\text{DQVI} \iff \dot{x} = \Psi(x) \equiv A(x, y(x)) + B(x, y(x))u(x, y(x)),$$

where $\Psi$ is Lipschitz continuous and piecewise analytic near $x_0$.

Therefore, $\exists$ a unique $C^1$ solution trajectory $x^*(t)$ passing through $x^*(t_0) = x_0$ for $t$ sufficiently near $t_0$; moreover, with

$$y^*(t) \equiv y(x(t)) \text{ and } u^*(t) \equiv u(x(t), y(x(t))),$$

$(x^*(t), y^*(t), u^*(t))$ solves the DQVI, and $(y^*(t), u^*(t))$ are Lipschitz continuous (albeit not yet proven piecewise analytic).
System Modes of the DQVI

Roughly, a mode of the DQVI is a resolution of the disjunctive complementary slackness condition into active and inactive constraints.

For our purpose here, a mode is determined by a tuple of index sets, \((\alpha, \beta, \gamma, I, J)\), with \((\alpha, \beta, \gamma)\) partitioning \(\{1, \cdots, m\}\) and \((I, J)\) partitioning \(\{1, \cdots, \ell\}\), respectively, resulting in a system of differential algebraic equations:

\[
\begin{align*}
\dot{x} &= A(x, y) + B(x, y)u \\
G_{\alpha}(x, y) &= 0 \quad [\leq y_{\alpha}] \\
G_{\beta}(x, y) &= 0 = y_{\beta} \\
y_{\gamma} &= 0 \quad [\leq G_{\gamma}(x, y)] \\
C(x, y) + N(x)u - E^T\lambda &= 0 \\
(H(x, y) + Eu)_I &= 0 \quad [\leq \lambda_I] \\
\lambda_J &= 0 \quad [\leq (H(x, y) + Eu)_J]
\end{align*}
\]

Note that there are only 5, not 6, index sets.
Mode Switchings and (Non-)Zeno States

Mode switchings near, i.e., shortly before and after, a given state is the main issue to be addressed. [Çamlibel and Schumacher established forward non-Zenoness for passive LCSs.]

Important in order to faithfully simulate a continuous-time solution trajectory by time-stepping methods that typically produce only numerical trajectories.

A non-Zeno state of a given trajectory is one such that there are only finitely mode switches, or equivalently, there is no mode switch in a small time interval before and after the state. This is a local property of the state in question.

Previously, Shen and Pang (2005) established that an LCS with $D$ being a $P$-matrix has no Zeno states.

Pang and Shen (2007) extended the result to an NCS showing that a strongly regular state is non-Zeno.

Shen and Pang (2007) extended the LCS result to the a $B$-singleton case.

Çamlibel, Pang, and Shen (2007) established the non-Zenoness of a CLS.

None of the existing results apply to the DQVI.
The Main Theorem

Notation: Given the solution trajectory \((x^*(t), y^*(t), u^*(t))\) of the DQVI, let
\[
\alpha(t) \equiv \{ i : y_i^*(t) > 0 = G_i(x^*(t), y^*(t)) \}, \\
\beta(t) \equiv \{ i : y_i^*(t) = 0 = G_i(x^*(t), y^*(t)) \}, \\
\gamma(t) \equiv \{ i : y_i^*(t) = 0 < G_i(x^*(t), y^*(t)) \}, \\
\mathcal{I}(t) \equiv \{ i : [H(x^*(t), y^*(t)) + Eu^*(t)]_i = 0 \}, \\
\mathcal{J}(t) \equiv \{ j : [H(x^*(t), y^*(t)) + Eu^*(t)]_i > 0 \}
\]

**Theorem.** Let \(t_0 > 0\) be a positive time (for forward and backward result). Under assumptions (A–D), there exist a scalar \(\varepsilon > 0\) and ten index sets, \((\alpha_\pm, \beta_\pm, \gamma_\pm)\) and \((\mathcal{I}_\pm, \mathcal{J}_\pm)\) such that
\[
(\alpha(t), \beta(t), \gamma(t), \mathcal{I}(t), \mathcal{J}(t)) \equiv \begin{cases} 
(\alpha_-, \beta_-, \gamma_-, \mathcal{I}_-, \mathcal{J}_-), & \forall t \in [t_0 - \varepsilon, t_0), \\
(\alpha_+, \beta_+, \gamma_+, \mathcal{I}_+, \mathcal{J}_+), & \forall t \in (t_0, t_0 + \varepsilon].
\end{cases}
\]
Moreover, the trajectory \((x^*(t), y^*(t), u^*(t))\) is analytic in the two open subintervals \((t_0 - \varepsilon, t_0)\) and \((t_0, t_0 + \varepsilon)\). \(\square\)
AVI reduction : positive definite \( N(x) \)

By means of linear algebraic operations, the AVI \((K(x,y), C(x,y), N(x))\) is equivalent to the AVI \(\left( \mathcal{P}_E(H^\perp(x,y)), \tilde{C}(x,y), \tilde{N}(x) \right)\), where

- the matrix \( \hat{E} \) has full column rank, and
- \( H^\perp(x,y) \in \text{Range}(E)^\perp \)
- \( H(x,y) + Eu = H^\perp(x,y) + \hat{E}\tilde{u} \), with
  
  \[ u \in \text{SOL}(K(x,y), C(x,y), N(x)) \text{ with multiplier } \lambda \]
  \[ \iff \tilde{u} \equiv \hat{H}(x,y) - \hat{M}(x)\tilde{C}(x,y) + \hat{M}(x)\hat{E}^T\lambda \]
  \[ \in \text{SOL}(\mathcal{P}_E(H^\perp(x,y)), \tilde{C}(x,y), \tilde{N}(x)) \text{ with same multiplier } \lambda \]

DVI is equivalent to

\[
\begin{align*}
\dot{x} &= \tilde{A}(x,y) + \tilde{B}(x,y)\tilde{u} \\
0 &\leq y \perp G(x,y) \geq 0 \\
\tilde{u} &= \text{SOL} \left( \mathcal{P}_E(H^\perp(x,y)), \tilde{C}(x,y), \tilde{N}(x) \right)
\end{align*}
\]

Purpose: \( \hat{H}(x,y) + \hat{E}\tilde{u} = 0 \iff [\tilde{u} = 0 \text{ and } \hat{H}^\perp(x,y) = 0] \)
**Transformed model functions.** Let

\( \hat{E} \equiv \) a submatrix of \( E \) whose columns form a basis of \( \text{Range}(E) \)

\[ E = \hat{E} \begin{bmatrix} I & F \end{bmatrix} \] for some matrix \( F \); \( M(x) \equiv N(x)^{-1} \)

\( \hat{M}(x) \equiv \begin{bmatrix} I & F \end{bmatrix} M(x) \begin{bmatrix} I \\ F^T \end{bmatrix} \); \( \hat{N}(x) \equiv \hat{M}(x)^{-1} \)

\( \hat{C}(x,y) \equiv \hat{N}(x) \begin{bmatrix} I & F \end{bmatrix} M(x) C(x,y) \)

\( \hat{H}(x,y) \equiv (\hat{E}^T \hat{E})^{-1} \hat{E}^T H(x,y) \)

\( H^\perp(x,y) \equiv \) orthogonal projection of \( H(x,y) \) onto \( \text{Range}(\hat{E})^\perp \)

\( \hat{B}(x,y) \equiv B(x,y) M(x) \begin{bmatrix} I \\ F^T \end{bmatrix} \hat{N}(x) \)

\( \tilde{A}(x,y) \equiv A(x,y) - B(x,y) \left\{ M(x) - M(x) \begin{bmatrix} I \\ F^T \end{bmatrix} \hat{N}(x) \begin{bmatrix} I & F \end{bmatrix} M(x) \right\} C(x,y) \)

\( -\hat{B}(x,y)\hat{H}(x,y) \)

\( \tilde{C}(x,y) \equiv \hat{C}(x,y) - \hat{N}(x)\hat{H}(x,y) \)
A glimpse at the inductive proof
induction on $m + \ell$ : the number of algebraic variables

Taking $t_0 = 0$, and argue only forward time.

Three easy cases:

(i) $y^0 \neq 0 \Rightarrow \exists i$ such that $G_i(x^*(t), y^*(t)) = 0 < y_i^*(t)$ for all $t > 0$ sufficiently small:

Apply implicit function theorem to $G_i(x, y) = 0$ to eliminate $y_i$; thus reducing $m$ by one.

(ii) $y^0 = 0 \neq G(x^0, y^0) \Rightarrow \exists i$ such that $y_i^*(t) = 0 < G_i(x^*(t), y^*(t))$ for all $t > 0$ sufficiently small:

Set $y_i = 0$ and remove the constraint $G_i(x, y) \geq 0$; again reducing $m$ by one.

(iii) $y^0 = G(x^0, y^0) = 0$ and $H(x^0, y^0) \neq 0$

$\Rightarrow \exists j$ such that $[H(x^*(t), y^*(t)) + Eu^*(t)]_j > 0$ for all $t > 0$ sufficiently small

Remove this constraint in $K(x, y)$, reducing $\ell$ by one.

The hard case: $y^0 = G(x^0, y^0) = 0$ and $H(x^0, y^0) = 0$.

Need expansion of trajectory near $t_0$ to continue the reduction!
Lie derivatives

For given analytic vector-valued functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^m$, 

$L_0^f \phi(x) \equiv \phi(x)$ and $L_j^f \phi(x) \equiv (JL_{j-1}^f \phi(x)) f(x)$, $j \geq 1$, 

where $JL_{f}^{j-1} \phi(x)$ denotes the Jacobian matrix of $L_{f}^{j-1} \phi(x)$.

Functional expansion in terms of Lie derivatives

If $\tilde{x}^f(t)$ denotes the unique solution of the ODE: 

$$\dot{x} = f(x), \; x(0) = x^0.$$

then for all $t > 0$ sufficiently small, 

$$\phi(\tilde{x}^f(t)) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{d^j \phi(\tilde{x}^f(0))}{dt^j} = \sum_{j=0}^{\infty} \frac{t^j}{j!} L_{j}^f \phi(x^0)$$

Let 

$$f(x) \equiv A(x, 0), \; g(x) \equiv G(x, 0), \; h(x) \equiv H(x, 0), \; \text{and} \; c(x) \equiv C(x, 0).$$
Another easy case

involving Lie derivatives

If

(a) $L^j_f g(x^0) = 0$ for all $j \geq 0$,
(b) $L^j_f h(x^0) = 0$ for all $j \geq 0$, and
(c) $c(\tilde{x}^f(t)) \in \text{pos}(E^T)$ for all $t \geq 0$,

then $(\tilde{x}^f(t), 0, 0)$ is the unique solution to the DQVI for all $t \geq 0$. Furthermore, for all $t \geq 0$,

$\alpha(t) = \gamma(t) = \mathcal{I}(t) = \emptyset$, $\beta(t) = \{1, \ldots, m\}$, $\mathcal{I}(t) = \{1, \ldots, \ell\}$. 
The solution expansion: the set up

Let \( \nu \geq 0 \) be an integer such that

(a) \( L_j^g(x^0) = 0 \) for all \( j = 0, \ldots, \nu - 1 \),

(b) \( L_j^h(x^0) = 0 \) for all \( j = 0, \ldots, \nu - 1 \), and

(c) \( \hat{c}(t) \equiv \sum_{j=0}^{\nu-1} \frac{t^j}{j!} L_j^c(x^0) \in \text{pos}(E^T) \) for all \( t \geq 0 \) sufficiently small.

Consider the linear complementarity problem (LCP) in the variable \( v \in \mathbb{R}^m \):

\[
0 \leq v \perp L_\nu^g(x^0) + J_y G(x^0, y^0) v \geq 0.
\]

Define four vectors \( v^*, a^*, b^* \), and \( \xi^* \):

- \( v^* \) is the (well-defined) unique solution of the above LCP;
- \( a^* \equiv L_\nu^c(x^0) + J_y C(x^0, 0) v^*; \)
- \( b^* \equiv L_\nu^h(x^0) + J_y H(x^0, 0) v^*; \)
- \( \xi^* \in \mathbb{R}^\ell \) is such that the unique solution to the AVI \( (b^* t^\nu/\nu!, \hat{c}(t) + t^\nu a^*/\nu!, N(x^0)) \) is given by \( \xi^* t^\nu/\nu! \) for all \( t \geq 0 \) sufficiently small;
The solution expansion

\[ x^*(t) = x^0 + \sum_{j=0}^{\nu} L^j f(x^0) \frac{t^{j+1}}{(j+1)!} + \]
\[ \frac{t^{\nu+1}}{(\nu + 1)!} \left\{ J_y A(x^0, 0) v^* + J_y \left[ B(x^0, \bullet) u^0 \right]_{y=0} v^* + B(x^0, 0) \xi^* \right\} + o(t^{\nu+1}), \]
\[ y^*(t) = \frac{t^{\nu}}{\nu!} v^* + o(t^{\nu}), \]
\[ u^*(t) = \frac{t^{\nu}}{\nu!} \xi^* + o(t^{\nu}), \]
\[ G(x^*(t), y^*(t)) = \frac{t^{\nu}}{\nu!} \left( L^\nu f(x^0) + J_y G(x^0, y^0) v^* \right) + o(t^{\nu}), \]
\[ H(x^*(t), y^*(t)) + Eu^*(t) = \frac{t^{\nu}}{\nu!} \left( b^* + E \xi^* \right) + o(t^{\nu}). \]

The key is the orders of the expansions.
Let $\nu$ satisfy the assumption in the solution expansion. Consider three cases:

(i) $L^\nu_f g(x^0) \neq 0$

(ii) $L^\nu_f h(x^0) \neq 0$, or

(iii) it is not true that $\hat{c}(t) + L^\nu_f c(x^0)t^{\nu}/\nu \in \text{pos}(E^T)$ for all $t > 0$ sufficient small.

The solution expansion allows the elimination of at least one algebraic variable, resulting in a reduced system that continues to satisfy the model assumptions (A–D).
Concluding remarks

Motivated by the application to the frictional contact problem, we have established the non-Zenoness of a class of DQVIs under a set of model assumptions, using advanced sensitivity results of finite-dimensional variational inequalities.

Presently open is the full analysis of the frictional contact problem under the quadratic Coulomb friction law!
Lecture II

Continuous-Time Dynamic User Equilibrium via Differential Complementarity Systems

Monday September 10, 2012, 11:30–12:30 PM
Outline of Presentation

• **Part A:** Continuous-time Dynamic User Equilibrium (DUE)
  — dynamic traffic modeling: a synopsis
  — a general predictive model

• **Part B:** A first simplification: single-bottleneck
  — LCS formulation and numerical solution
  — collaborators: Lanshan Han, Satish Ukkusuri, and Ramadurai Gitakrishnan

• **Part C:** A point-queue approach to model travel times
  — Vickrey’s model and its modifications
  — collaborators: Xueguang (Jeff) Ban (and student Ma Rui) and Henry Liu

• **Part D:** The Instantaneous Dynamic User Equilibrium (IDUE) Problem
  — sketch of model
  — same collaborators as Part III
Part A

Continuous-time Dynamic User Equilibrium (DUE)
Dynamic Traffic Modeling: A Synopsis

- Traffic evolves over time and space.

- Static models assuming steady traffic states cannot capture such temporal and spatial evolution properly.

- User equilibrium route choice, described by Wadrop’s principle, and traffic flow propagation are 2 central ingredients in dynamic traffic modeling.

- Delays caused by congestion should be included in models.

- Most “dynamic” traffic models are formulated in discrete time.

- Scattered continuous-time models exist but are at best minimally analyzed.

- Ad-hoc modifications and explicit discretization schemes are the prevailing practice in treating continuous-time models.

- There is no proper mathematical framework that can capture both aspects of DUE, choice behavior and system dynamics, until recently.

- There are 2 ways to model flow propagation: predictive (forward delays) versus instantaneous (backward delays).
**DVI formulation of DUE**: Notations

**Network notation:**
- \( \mathcal{N}/\mathcal{L} \): node/link set
- \( \mathcal{W} \): set of origin-destination (OD) pairs, a subset of \( \mathcal{N} \times \mathcal{N} \)
- \( \mathcal{S} \): set of destinations, a subset of \( \mathcal{N} \)
- \( \mathcal{P}_w \): set of paths joining OD pair \( w \in \mathcal{W} \)
- \( d_w \): (fixed) travel demand between OD pair \( w \in \mathcal{W} \), assumed positive
- \( \mathbf{d} \triangleq (d_w)_{w \in \mathcal{W}} \): vector of travel demands
- \( \mathcal{P} \triangleq \bigcup_{w \in \mathcal{W}} \mathcal{P}_w \): set of all paths
- \( \mathcal{N}_s \): set of nodes \( i \neq s \) for which there is a path in \( \mathcal{P} \) joining \( i \) to destination \( s \in \mathcal{S} \)
- \( \mathcal{L}_s \): set of links contained in paths in \( \mathcal{P} \) that join nodes in \( \mathcal{N}_s \) to destination \( s \in \mathcal{S} \) \((\mathcal{L}_{s'} \cap \mathcal{L}_s \neq \emptyset \) is possible for \( s' \neq s \))
  (with no loss of generality, we assume that \( \bigcup_{s \in \mathcal{S}} \mathcal{L}_s = \mathcal{L} \));
- \( \mathcal{S}_{ij} \): subset of destinations \( s \in \mathcal{S} \) such that the link \((i, j) \in \mathcal{L}_s \)
- \( \overline{C}_{ij} \): (given) flow capacity on link \((i, j) \)
General model:

Time-dependent variables

- $x_{ij}^s(t)$: the amount of vehicular flow on link $(i, j)$ destined for node $s \in S$
- $x(t) \triangleq (x_{ij}^s(t))_{s \in S, (i,j) \in \mathcal{L}}$
- $p_{ij}^s(t)$: rate of entry flow on link $(i, j)$ destined for node $s \in S$
- $v_{ij}^s(t)$: rate of exit flow on link $(i, j)$ destined for node $s \in S$ on link $(i, j)$
- $\tau_{ij}(t)$: actual travel time on link $(i, j)$
- $\eta_i^s(t)$: the minimum travel time from node $i$ to destination $s$ at time $t$

Point-queue model: (to be introduced in Part C)

- $\tau_{ij}^0$: (given) free flow time on link $(i, j)$
- $q_{ij}^s(t)$: queue length of traffic on link $(i, j)$ destined for node $s \in S$
The Delay DVI

For all $s \in S$ and for almost all $t \in (0, T]$,

$$
\dot{x}_{ij}^s(t) = p_{ij}^s(t) - v_{ij}^s(t) \quad \text{mass balance dynamics}
$$

$$
\dot{\tau}_{ij}(t) = \frac{p_{ij}^s(t)}{v_{ij}^s(t + \tau_{ij}(t))} - 1 \quad \text{FIFO preserving flow propagation}
$$

forward delay

$$
0 \leq p_{ij}^s(t) \perp \tau_{ij}(t) + \eta_j^s(t + \tau_{ij}(t)) - \eta_i^s(t) \geq 0 \quad \text{Waldrop's route choice}
$$

same delay

$$
0 \leq \eta_i^s(t) \perp \sum_{j:(i,j) \in \mathcal{L}} p_{ij}^s(t) - \sum_{k:(k,i) \in \mathcal{L}} v_{ki}^s(t) - d_i^s(t) \geq 0 \quad \text{flow conservation}
$$

$$
\tau_{ij}(t) = \tau_{ij}(x(t)) \quad \text{link travel time function}
$$

$$
x_{ij}^s(0) = \xi_{ij}^s \quad \text{initial condition}.
$$

A very difficult problem:

state-dependent forward time-delay differential variational system!
Part B

A first simplification: single bottleneck

A reference

The Continuous-Time Single-Bottleneck Model

- Vehicles traverse from an origin to a destination connected by a **single link**, and therefore no route choice is involved;
- There is a single bottleneck on the link that determines the capacity of the link (s vehicles per unit time);
- The travel time includes two parts: the fixed travel time for traversing the link $T_0$ (can be assumed to be 0) and the queue delay $T_q$;
- The queue delay is determined by a **point-queue model**, in which vehicles are approximated by a fluid flow (non-atomic setting) and have no physical lengths;
- Vehicles have a desired arrival time; the total cost thus includes three parts: travel cost, early arrival penalty, and late arrival penalty.
- We consider a continuous-time multi-user class setting; each class has its own desired arrival time, unit costs, and demand.

- **Equilibrium condition (Wardrop’s principle)**: whenever there is a positive departure rate, the cost of users departing at that time is the same and is less than the cost that would be experienced by a user departing at a time with zero departure rate
Some notations

Parameters

- $G$: user classes with elements $g \in G$
- $T$: total time duration
- $d_g$: total travel demand of class $g$
- $s$: bottleneck capacity (number of vehicles per unit time)
- $\alpha_g$: class $g$’s value of travel time ($ per unit time$)
- $\beta_g$: class $g$’s value of schedule delay when arriving early ($ per unit time$)
- $\gamma_g$: class $g$’s value of schedule delay when arriving late ($ per unit time$)
- $t^*_g$: class $g$’s preferred arrival time at destination

Primary variables (all nonnegative)

- $TT(t)$: queue time of users departing at time instance $t$
  $TT(t) + t$ is the arrival time of users departing at time $t$
- $N_g(t)$: class $g$’s cumulative departures in time interval $[0,t]$
- $c^*_g$: equilibrium cost of class $g$
Derived variables

\( u(t) \) slack variable \( \triangleq \frac{dT(t)}{dt} - \frac{1}{s} \left[ \sum_{g \in G} r_g(t) - s \right] \)

entry flow rate

\( r_g(t) \) class \( g \)'s departure rate at time \( t \) (departures per unit time)
\( \triangleq \frac{dN_g(t)}{dt} \)

\( e_g(t) \) duration between early arrival and preferred arrival time \( t^*_g \) of users in class \( g \) \( \triangleq \max \{ 0, t^*_g - (TT(t) + t) \} \)

\( \ell_g(t) \) duration between late arrival and preferred arrival time \( t^*_g \) of users in class \( g \) \( \triangleq \max \{ 0, (TT(t) + t) - t^*_g \} \)

\( C'_g(t) \) travel cost of users in class \( g \) departing at time \( t \)
\( \triangleq \alpha_g TT(t) + \beta_g e_g(t) + \gamma_g \ell_g(t) \)
Equilibrium conditions

- The departure rate is nonnegative;
- The total cost for departures at any time is greater than or equal to the equilibrium cost, and equality holds when the departure rate is positive,
  \[ 0 \leq r_g(t) \perp C_g(t) - c_g^* \geq 0, \quad \forall g \in G. \]

Travel demand

- The time derivative of the cumulative departure is the departure rate:
  \[ \frac{dN_g(t)}{dt} = r_g(t), \quad \forall g \in G; \]
- For each class, all vehicles depart from the origin in \([0;T]\):
  \[ N_g(T) = d_g, \quad \forall g \in G. \]

Initial conditions

- At time 0, the equilibrium condition holds:
  \[ 0 \leq N_g(0) \perp C_g(0) - c_g^* \geq 0, \quad \forall g \in G; \]
- The travel time at time 0 is given by \(TT(0) = \max \left( 0, \frac{1}{s} \sum_{g \in G} N_g(0) - 1 \right). \)
The LCS formulation for the single-bottleneck model

(A) For almost all \( t \in [0,T] \):

\[
\frac{dT(t)}{dt} = u(t) + \frac{1}{s} \left( \sum_{g \in G} r_g(t) - s \right)
\]

\[
\frac{dN_g(t)}{dt} = r_g(t), \quad \forall g \in G
\]

\[0 \leq u(t) \perp TT(t) \geq 0\]

\[0 \leq r_g(t) \perp C_g(t) - c^*_g \geq 0, \quad \forall g \in G,\]

(B) the initial conditions:

\[TT(0) = \max \left( 0, \frac{1}{s} \sum_{g \in G} N_g(0) - 1 \right)\]

\[0 \leq N_g(0) \perp C_g(0) - c^*_g \geq 0, \quad \forall g \in G,\]

(C) the boundary conditions: \( N_g(T) = d_g \) for all \( g \in G \).
Convergence via numerical time-stepping

Main Theorem. There exist
— nonnegative scalars \( \{c_g^\ast\}_{g \in G} \)
— absolutely continuous functions \( TT(t) \) and \( \{N_g(t)\}_{g \in G} \)
— integrable functions \( \{r_g(t)\}_{g \in G} \) and \( u(t) \)
so that conditions (A)–(C) are all satisfied.

Proof by construction:

• Divide the time interval \([0, T]\) into \( \nu \) subintervals of equal length \( h_\nu \triangleq T/\nu \).
• Computing the nonnegative scalars \( \{c_g^{\nu,\ast}\}_{g \in G} \) and the discrete-time iterates
  \( \{TT^{\nu,0}, TT^{\nu,1}, \ldots, TT^{\nu,\nu}\}, \ \{N_g^{\nu,0}, N_g^{\nu,1}, \ldots, N_g^{\nu,\nu}\}_{g \in G}, \ \{r_g^{\nu,1}, \ldots, r_g^{\nu,\nu}\}_{g \in G} \).
• Construct numerical trajectories on \([0, T]\)
  — continuous and piecewise linear \( \hat{TT}^{\nu}(t), \hat{N}_g^{\nu}(t) \)
  — piecewise constant functions \( \hat{u}^{\nu}(t) \) and \( \hat{r}^{\nu}(t) \).
(A\(_{\nu}\)) For all \(i = 1, \ldots, \nu,\)

\[
0 \leq TT^{\nu,i} \downarrow TT^{\nu,i} - TT^{\nu,i-1} - \frac{h\nu}{s} \left[ \sum_{g \in G} r^{\nu,i}_g - s \right] \geq 0
\]

\[
N^{\nu,i}_g - N^{\nu,i-1}_g = h r^{\nu,i}_g, \quad \forall g \in \mathbb{G}
\]

\[
0 \leq r^{\nu,i}_g \downarrow - \frac{\gamma_g (t^*_g - i h\nu)}{\alpha_g + \gamma_g} + TT^{\nu,i} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g} \max \left\{ 0, t^*_g - (TT^{\nu,i} + i h\nu) \right\} - \frac{c^{\nu,*}_g}{\alpha_g + \gamma_g} \geq 0, \quad \forall g \in \mathbb{G},
\]

(B\(_{\nu}\)) the initial conditions:

\[
0 \leq TT^{\nu,0} \downarrow TT^{\nu,0} - \left[ \frac{1}{s} \sum_{g \in \mathbb{G}} N^{\nu,0}_g - 1 \right] \geq 0
\]

\[
0 \leq N^{\nu,0}_g \downarrow - \frac{\gamma_g t^*_g}{\alpha_g + \gamma_g} + TT^{\nu,0} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g} \max \left\{ 0, t^*_g - TT^{\nu,0} \right\} - \frac{c^{\nu,*}_g}{\alpha_g + \gamma_g} \geq 0, \quad \forall g \in \mathbb{G}
\]

(C\(_{\nu}\)) the boundary conditions: \(N^{\nu,\nu}_g = d_g\) for all \(g \in \mathbb{G}.\)
The time-discretized $\text{LCP}_\nu$

Substituting the boundary equations:

\[
0 \leq TT^{\nu,0} \perp TT^{\nu,0} + \frac{h_\nu}{s} \sum_{g \in G} \sum_{i=1}^{\nu} r^{\nu,i}_g - \frac{1}{s} \sum_{g \in G} d_g + 1 \geq 0
\]

\[
0 \leq TT^{\nu,i} \perp TT^{\nu,i} - TT^{\nu,i-1} - \frac{h_\nu}{s} \sum_{g \in G} r^{\nu,i}_g + h_\nu \geq 0, \quad i = 1, \cdots, \nu
\]

\[
0 \leq r^{\nu,i}_g \perp \frac{i h_\nu \gamma_g}{\alpha_g + \gamma_g} + TT^{\nu,i} - TT^{\nu,0} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g} (e^{\nu,i}_g - e^{\nu,0}_g) + \frac{f^{\nu}_g}{\alpha_g + \gamma_g} \geq 0,
\]

\[
i = 1, \cdots, \nu; \quad g \in G
\]

\[
0 \leq f^{\nu}_g \perp d_g - h_\nu \sum_{i=1}^{\nu} r^{\nu,i}_g \geq 0, \quad g \in G
\]

\[
0 \leq e^{\nu,i}_g \perp -(t^*_g - i h_\nu) + TT^{\nu,i} + e^{\nu,i}_g \geq 0, \quad i = 0, 1, \cdots, \nu; \quad g \in G.
\]

- The $\text{LCP}_\nu$ has a solution that can be computed by Lemke's algorithm with any positive vector as the covering vector.
Part C

Traffic loading: point-queue models

Two references


In 1969, William Vickrey proposed an ordinary differential equation (ODE) in explicit form to describe the dynamics of the vehicular flows in a congested traffic network modeled as a point queue with no physical length.

Widely researched in the traffic literature, this dynamical system is known to generate negative queue lengths.

While ad hoc discrete-time schemes have been proposed to remedy this drawback, there is no continuous-time formulation of queue-length dynamics that ensures this nonnegativity property.

Most importantly, there is a total void in the rigorous analysis of continuous-time dynamic user equilibrium (DUE) problems. Such models exist but are immediately converted into a discrete-time setting with no follow-up analysis of the interrelations.

Our research provides a first step in this direction by proposing a continuous-time queue-length dynamics that remedies the drawback in Vickrey’s original model (this presentation). A subsequent study employs the new point-queue model to study the continuous-time instantaneous DUE (not presented here).
Point-Queue Models for Travel-Time Prediction

a given origin-destination pair on a single link

Notation

- $\tau_0$ (given) free flow time on link
- $p(t)$ entry flow at time $t$
- $q(t)$ queue length of traffic at time $t$
- $\tau(t)$ actual travel time
- $C$ link capacity
- $v(t)$ exit flow at time $t$

Vickrey’s original model: with $p(t - \tau_0)$ known at time $t$,

(a) the dynamics of the queue is: 
\[
\dot{q}(t) = \begin{cases} 
0 & \text{if } t \in (0, \tau_0) \\
 p(t - \tau_0) - v(t) & \text{if } t > \tau_0 
\end{cases} 
\]

(b) the initial queue is zero: $q(\tau_0) = 0$;

(c) for $t < \tau_0$, the exit flow rate is: $v(t) = 0$;

(d) for $t > \tau_0$, the exit flow rate is: 
\[
v(t) = \begin{cases} 
\min (C, p(t - \tau_0)) & \text{if } q(t) = 0 \\
C & \text{if } q(t) \neq 0 
\end{cases} 
\]

(e) for $t \in [0, T - \tau_0]$, $\tau(t) = \tau_0 + (C)^{-1}q(t + \tau_0)$. 

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Substitution yields the nomsMOOTH queue-length dynamics: for \( t > \tau_0 \),

\[
\dot{q}(t) = \begin{cases} 
\max \left( p(t - \tau_0) - C, 0 \right) & \text{if } q(t) = 0 \\
p(t - \tau_0) - C & \text{if } q(t) \neq 0 
\end{cases},
\]

where the right-hand side is discontinuous in the state variable \( q \) at a time \( t \) such that \( p(t - \tau_0) < C \).

**Time discretizations**

Choose a step size \( h > 0 \) and partition the time interval \([\tau_0, T]\) into \( N_h + 1 \triangleq T/h \) time steps, each of equal length \( h \):

\[
\tau_0 \triangleq t_{h,0} < t_{h,1} < \cdots < t_{h,N_h} < t_{h,N_h+1} \triangleq T,
\]

where \( t_{h,r} \triangleq \tau_0 + rh \) for \( r = 1, \cdots, N_h \). A discretization scheme computes the discrete-time iterates \( \{q^{h,r}\}_{r=1}^N \) with \( \hat{p}^{h,r+1} \triangleq p(t_{h,r+1} - \tau_0) \) and \( q^{h,r} \approx q(t_{h,r}) \).

- \( q^{h,r+1} = q^{h,r} + h \left\{ \begin{array}{ll} 
\max \left( \hat{p}^{h,r+1} - C, 0 \right) & \text{if } q^{h,r} = 0 \\
\hat{p}^{h,r+1} - C & \text{if } q^{h,r} \neq 0 
\end{array} \right. 
\)
  the explicit scheme

- \( q^{h,r+1} = q^{h,r} + h \left\{ \begin{array}{ll} 
\max \left( \hat{p}^{h,r+1} - C, 0 \right) & \text{if } q^{h,r+1} = 0 \\
\hat{p}^{h,r+1} - C & \text{if } q^{h,r+1} \neq 0 
\end{array} \right. 
\)
  the implicit scheme
Deficiencies and remedies of the discretized Vickrey models

• for both schemes, \( q^{h,r+1} < 0 \) if \( q^{h,r} \neq 0 \) but \( q^{h,r} + h(\hat{p}^{h,r+1} - \overline{C}) < 0 \);

• for the implicit scheme, \( q^{h,r+1} \) does not exist if \( q^{h,r} \neq 0 = q^{h,r} + h(\hat{p}^{h,r+1} - \overline{C}) \) and \( \hat{p}^{h,r+1} < \overline{C} \).

A natural modification by Nie-Zhang (2005):

\[
q^{h,r+1} \triangleq \max \{ 0, q^{h,r} + h(\hat{p}^{h,r+1} - \overline{C}) \}.
\]

Is there a continuous-time dynamics \(\Rightarrow\) the NZ discrete-time expression? Yes!

The 2011 Pang-Han-Ramadurai-Ukkusuri (PHRU) single-bottleneck model:

\[
\dot{q}(t) = u(t) + p(t - \tau_0) - \overline{C} \\
0 \leq u(t) \perp q(t) \geq 0,
\]  

which is a linear complementarity system (LCS), or an ODE in implicit form:

\[
\min \left( q(t), \dot{q}(t) + \overline{C} - p(t - \tau_0) \right) = 0.
\]

Is there a “smooth” dynamics for traffic queue lengths? Yes!
Queue length properties

In continuous time,

(a) $q(t)$ is absolutely continuous and nonnegative on $[\tau_0, T]$; thus $\dot{q}(t)$ is well defined for almost all $t \in [\tau_0, T]$;

(b) the differential system holds at almost all $t \in [\tau_0, T]$;

(c) the derived exit flow $v(t) \triangleq p(t - \tau_0) - \dot{q}(t)$, which is well defined only for almost all times, is nonnegative at almost all $t \in [\tau_0, T]$;

(d) the FIFO property, i.e., $\dot{\tau}(t) \geq -1$, holds at almost all $t \in [\tau_0, T]$.

In discrete time, or for a discretized model:

(a) $q^{h,r+1} \geq 0$; $v^{h,r+1} \geq 0$, and

(b) $\tau^{h,r+1} + h \geq \tau^{h,r}$.

The LCS point-queue model has all these properties.
A smooth dynamics: the $\alpha$-model

For $\alpha > 0$ let

$$\dot{q}(t) = \max\left(p(t - \tau_0) - \bar{C}, -\alpha q(t)\right).$$

(3)

Comparisons of the V(ickrey)-model, the LCS-model, and the $\alpha$-model

• If $q(t) = 0$, then both the V-model (1) and the $\alpha$-model (3) postulate that

$$\dot{q}(t) = \max\left(p(t - \tau_0) - \bar{C}, 0\right),$$

whereas the LCS-model (2) merely postulates that $\dot{q}(t) \geq p(t - \tau_0) - \bar{C}$.

• If $q(t) \neq 0$, then both the V-model (1) and the LCS-model (2) postulate that

$$\dot{q}(t) = p(t - \tau_0) - \bar{C};$$

The $\alpha$-model (3) postulates the same expression for $\dot{q}(t)$ if in addition

$$\alpha q(t) + p(t - \tau_0) - \bar{C} \geq 0$$

• If $q(t) \neq 0$ and $\alpha q(t) + p(t - \tau_0) - \bar{C} < 0$, then the $\alpha$-model (3) postulates that

$$\dot{q}(t) = -\alpha q(t) > p(t - \tau_0) - \bar{C},$$

where the last term is what both (1) and (2) postulate in this case.
Discretization of the $\alpha$-model

\[ q^{h,r+1} = \begin{cases} \max \left\{ \left( 1 - h \alpha \right) q^{h,r}, q^{h,r} + h \left( \hat{p}^{h,r+1} - \overline{C} \right) \right\}, & \text{the explicit scheme} \\ \max \left\{ \frac{1}{1 + h \alpha} q^{h,r}, q^{h,r} + h \left( \hat{p}^{h,r+1} - \overline{C} \right) \right\}, & \text{the implicit scheme.} \end{cases} \]

By linear interpolation, construct the numerical queue-length trajectory from the implicit scheme: for \( r = 0, 1, \ldots, N_h \), let

\[ \hat{q}^h(t) \triangleq q^{h,r} + \frac{q^{h,r+1} - q^{h,r}}{h} (t - t_{h,i}), \quad \text{for } t \in [t_{h,r}, t_{h,r+1}]. \]

**Inflow assumption (A):** the inflow rate \( p(t - \tau_0) \) is a nonnegative, bounded, integrable function on \((\tau_0, T]\) equal to the weak limit of piecewise constant functions \( \hat{p}^h(t) \) as \( h \downarrow 0 \), which satisfy the following two conditions:

(i) for all \( h > 0 \) and all \( r = 0, 1, \ldots, N_h \),

\[ \hat{p}^h(t) = \hat{p}^{h,r+1} \triangleq p(h_{r+1} - \tau_0), \quad \forall t \in (t_{h,r}, t_{h,r+1}], \]

(ii) for every continuous function \( \phi \) on \([\tau_0, T]\),

\[ \lim_{h \to 0} \int_{\tau_0}^{T} \hat{p}^h(t) \phi(t) \, dt = \int_{\tau_0}^{T} p(t - \tau_0) \phi(t) \, dt. \]
Convergence Theorem.

• ∃ a sequence of step sizes \{h_\nu\} ↓ 0 and an absolutely continuous function \( \hat{q}(t) \) on \([\tau_0, T]\) such that \( \hat{q}^{h_\nu} \rightarrow \hat{q} \) uniformly on \([\tau_0, T]\).

• Any such limit function \( \hat{q}(t) \) is a weak solution of the ODE (3) in the sense that for any two times \( t_2 > t_1 \) in the interval \([\tau_0, T]\),

\[
\hat{q}(t_2) = \hat{q}(t_1) + \int_{t_1}^{t_2} \left[ \max \left( p(s - \tau_0) - \bar{C}, -\alpha \hat{q}(s) \right) \right] ds;
\]

or equivalently, the ODE (3) holds for almost all \( t \in (\tau_0, T] \). □

• The desired queue-length properties hold; i.e., nonnegativity and FIFO.

• Similar convergence can be established for the LCS queue-length model.
The $\alpha$-model as an LCS

\[
\dot{q}^\alpha(t) = u^\alpha(t) + p(t - t_0) - \bar{C}
\]
\[
0 \leq u^\alpha(t) \perp \frac{1}{\alpha} \dot{q}^\alpha(t) + q^\alpha(t) \geq 0.
\]

(5)

Asymptotic relation. Let the inflow rate $p(t - t_0)$ satisfy assumption (A). Let $(q^\alpha(t), u^\alpha(t))$ and $(q(t), u(t))$ be weak solutions of the $\alpha$-LCS (5) and the LCS-model (2), respectively. For every $t \in [\tau_0, T]$,

\[
\lim_{\alpha \to \infty} q^\alpha(t) = q(t).
\]

(6)

Summary of models

Vickrey $\implies$ LCS-model $\implies$ α-model $\implies$ LCS-model.
thing modification

“smooth” ODE $\alpha \to \infty$
Part D

The Instantaneous Dynamic User Equilibrium Problem
A point-queue based model on a network
An extension to multi-destinations adding the superscript “s”

Key consideration: How to distribute the exit and entry flows on links to destination-dependent flows?

Need to respect:

• sign restrictions of queue lengths, entry and exit flows
• the FIFO principle
• some fairness, or proportionality in destination-specific assignments in link capacities, entry and exit flows.

The FIFO preserving flow propagation implies:

(a) destination-independent ratios of inflow and outflow; i.e., for a link \((i,j) \in \mathcal{L}\), the entry-exit flow ratio

\[
\frac{p^s_{ij}(t)}{v^s_{ij}(t + \tau_{ij}(t))} = \dot{\tau}_{ij}(t) + 1, \quad \forall s \in S_{ij}
\]
depends only on the link \((i,j)\) and are independent of the destination \(s\); this implies that for any \(s \in S_{ij}\),

\[
\frac{p_{ij}^s(t)}{v_{ij}^s(t + \tau_{ij}(t))} = \frac{\sum_{s' \in S_{ij}} p_{ij}^{s'}(t)}{\sum_{s' \in S_{ij}} v_{ij}^{s'}(t + \tau_{ij}(t))} \cdot \frac{1}{|S_{ij}|} \sum_{s' \in S_{ij}} \frac{p_{ij}^{s'}(t)}{v_{ij}^{s'}(t + \tau_{ij}(t))} = \dot{\tau}_{ij}(t) + 1;
\]

(b) the FIFO property: for all \((i,j) \in \mathcal{L}\), \(\dot{\tau}_{ij}(t) + 1 \geq 0\) at all times \(t\) where \(\dot{\tau}(t)\) exists; equivalently, \(\tau_{ij}(t) + t\) is a non-decreasing function of \(t\).

Introducing the inverse link travel time \(g_{ij}(t)\), which is the travel time of vehicles exiting at time \(t\), we set

\[
v_{ij}^s(t) = \begin{cases} 
\frac{p_{ij}^s(t - g_{ij}(t))}{p_{ij}(t - g_{ij}(t))} v_{ij}(t) & \text{if } p_{ij}(t - g_{ij}(t)) > 0 \\
\text{known at time } t \\
0 & \text{if } p_{ij}(t - g_{ij}(t)) = 0,
\end{cases}
\]
where \( t' \triangleq t - g_{ij}(t) \) is the entry time of vehicles exiting at time \( t \) such that
\[
t' + \tau_{ij}(t') = t, \text{ or equivalently } t' + \tau^0_{ij} + \frac{q_{ij}(t' + \tau^0_{ij})}{C_{ij}} = t.
\]

The IDUE principle assumes that travelers make their route choice decision based on the prevailing traffic condition (i.e. what is happening now) instead of actual predictive traffic condition:
\[
0 \leq p^s_{ij}(t) \perp \underbrace{c_{ij}(t)}_{\text{instantaneous travel time}} + \underbrace{\eta^s_j(t)}_{\text{no delay}} - \eta^s_i(t) \geq 0,
\]
where \( c_{ij}(t) \triangleq \tau^0_{ij} + \frac{q_{ij}(t)}{C} \); in contrast to the predictive DUE:
\[
0 \leq p^s_{ij}(t) \perp \underbrace{\tau_{ij}(t)}_{\text{actual travel time}} + \underbrace{\eta^s_j(t + \tau_{ij}(t))}_{\text{time-dependent delay}} - \eta^s_i(t) \geq 0.
\]
The Instantaneous Dynamic User equilibrium model: complete formulation

- (Point-queue model to approximate delay function)
  
  \[ \dot{q}_{ij}(t) = \begin{cases} 
  0 & \text{if } t \in (0, \tau_{ij}^0) \\
  \max\left( p_{ij}(t) - \tau_{ij}^0 - \bar{C}_{ij}, -\alpha_{ij} q_{ij}(t) \right) & t > \tau_{ij}^0;
  \end{cases} \]

- (Equilibrium conditions)
  
  \[ 0 \leq p_{ij}^s(t) \perp \tau_{ij}^0 + \frac{q_{ij}(t)}{C_{ij}} + \eta_j^s(t) - \eta_i^s(t) \geq 0, \quad \forall (i, j) \in \mathcal{L}_s \]

  \[ 0 \leq \eta_i^s(t) \perp \sum_{j: (i, j) \in \mathcal{L}_s} p_{ij}^s(t) - \sum_{j: (j, i) \in \mathcal{L}_s} v_{ji}(t) - d_i^s(t) \geq 0, \quad \forall i \in \mathcal{N}_s \]

  \[ 0 = \eta_s^s(t), \]
Concluding Remarks

• DVIIs and DCSs are new mathematical paradigms that combine ODEs with MP, most suitable to model engineering and economic systems undergoing time evolution and mode changes.

• We have presented applications to the modeling of dynamic traffic equilibrium; the general predictive model remains highly challenging.

• Initial attempts based on prevailing traffic prediction lead to tractable formulations that can be solved by numerical time stepping and linear complementarity.
Lecture III

A Unified Numerical Scheme for Linear-Quadratic Optimal Control with Mixed State-Control Constraints

Tuesday September 11, 2012, 3–4 PM

Joint with Kanat Camlibel (Grönnigen), Lanshan Han (Illinois), and Maurice Heemels (Eindhoven).

• An application of Differential Variational Inequalities.
Contents of Presentation

- A review of finite-dimensional convex quadratic programs — the workhorse
- Statement of the control problem under study
- The assumptions — comments and key consequences
- The main theorem
- The unified time-stepping algorithm — derivation — convergence
Consider the convex QP \((Z(b), e, M)\) with 
\[ Z(b) \triangleq \{ u \in \mathbb{R}^n \mid b + Eu \geq 0 \} \]:

\[
\minimize_{z \in Z(b)} e^T z + \frac{1}{2} z^T M z, \quad (M \in \mathbb{R}^{n \times n} \text{ is symmetric positive semidefinite})
\]

Let \(\text{SOL}(Z(b), e, M)\) denote the optimal solution set (possibly empty).

(Existence of solutions). If \(Z(b) \neq \emptyset\), a necessary and sufficient condition for \(\text{SOL}(Z(b), e, M) \neq \emptyset\) is that \(e^T d \geq 0\) for all \(d \in Z(b)_{\infty} \cap \ker(M)\), where 
\[ Z(b)_{\infty} \triangleq \{ u \in \mathbb{R}^n \mid Eu \geq 0 \} \]

is the recession cone of \(Z(b)\).

(Characterization of optimality). A vector \(z \in \text{SOL}(Z(b), e, M)\) if and only if

\[
\exists \lambda \text{ such that } \begin{cases} 
0 = e + Mz - E^T \lambda \\
0 \leq \lambda \perp Ez + b \geq 0 
\end{cases} \text{ complementary slackness}
\]

(Boundedness of solutions). If \(Z_{\infty} \cap \ker(M) = \{0\}\), then \(\text{SOL}(Z(b), e, M)\) is nonempty for all \((e, b)\) for which \(Z(b) \neq \emptyset\). In this case, \(\bigcup_{b : Z(b) \neq \emptyset} \text{SOL}(Z(b), e, M)\) is bounded.

(Constancy of optimal gradients). \(M \text{SOL}(Z(b), e, M)\) is at most a singleton.
The linear-quadratic optimal control problem

Given

- a time horizon $T > 0$ and an initial state $\xi \in \mathbb{R}^n$
- matrices $S \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{\ell \times n}$, and $D \in \mathbb{R}^{\ell \times m}$, and
  \[
  \begin{bmatrix} P & Q \\ QT & R \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}; \text{ vectors } c \in \mathbb{R}^n \text{ and } f \in \mathbb{R}^\ell
  \]
- vector functions $p, r : [0, T] \to \mathbb{R}^n$ and $q : [0, T] \to \mathbb{R}^m$

find time-dependent vector functions $x : [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}^m$ to

\[
\begin{aligned}
\text{minimize } V(x, u) & \equiv c^T x(T) + \frac{1}{2} x(T)^T S x(T) + \\
& \int_0^T \left[ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} P & Q \\ QT & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right] dt \\
\text{subject to } x(0) &= \xi \\
\text{and } &\text{ for almost all } t \in [0, T]: \\
\dot{x}(t) &= r(t) + Ax(t) + Bu(t) \text{ and } f +Cx(t) + Du(t) \geq 0
\end{aligned}
\]

4
**Assumptions**

(A) (Convex but not strongly) the matrices $S$ and $\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$ are symmetric positive semidefinite;

(B) (Lipschitz inputs) the functions $p$, $q$, and $r$ are Lipschitz continuous on $[0, T]$;

(C) (a feasibility condition) a $C^1$ function $\hat{x}_{fs} : [0, T] \to \mathbb{R}^n$ with $\hat{x}_{fs}(0) = \xi$ and a continuous function $\hat{u}_{fs} : [0, T] \to \mathbb{R}^m$ exist such that for all $t \in [0, T]$:

$$\frac{d\hat{x}_{fs}(t)}{dt} = A\hat{x}_{fs}(t) + B\hat{u}_{fs}(t) + r(t) \text{ and } f + C\hat{x}_{fs}(t) + D\hat{u}_{fs}(t) \geq 0;$$

(D) (a primal condition) $[Ru = 0, Du \geq 0]$ implies $u = 0$;

(E) (a dual condition) $[D^T\mu = 0, \mu \geq 0]$ implies $(CA^iB)^T\mu = 0$ for all nonnegative integers $i$. 
Comments

(a) Due to the positive semidefiniteness, and not positive definiteness, of
\[
\begin{bmatrix}
P & Q \\
Q^T & R
\end{bmatrix},
\]
the integrand cannot easily be made a separable function of \((x, u)\) without additional algebraic constraints.

(b) Lipschitz inputs are needed for Lipschitz solutions.

(c) The polyhedron \(U(x) \triangleq \{u \mid f + Cx + Du \geq 0\}\) may be empty or unbounded for some \(x\).

(d) The **feasibility** assumption (C) is a departure from the existing literature wherein the existence of an optimal solution with certain smoothness properties is assumed.

(e) Assumption (D) ensures that whenever \(Z(b) \triangleq \{u \mid b + Du \geq 0\} \neq \emptyset\), the finite-dimensional (convex) quadratic program:
\[
\begin{align*}
\text{minimize} \quad & e^T z + \frac{1}{2} z^T R z \\
\text{subject to} \quad & z \in Z(b)
\end{align*}
\]
has an optimal solution.
Some sufficient conditions:

\[ D \text{ has full row rank} \Rightarrow [ D^T \mu = 0 \Rightarrow (CA^iB)^T \mu = 0 \text{ for all } i ] \]

\[ \Downarrow \]

\[ [(D^T \mu = 0, \mu \geq 0) \Rightarrow \mu = 0] \Rightarrow (E) \]

Main consequence: A perturbed Hoffman error bound.

Under (E), for any family of matrices:

\[ \begin{cases} 
B(h) \triangleq hB + \sum_{i=1}^{\infty} b_i h^{i+1} A^i B | h \geq 0 
\end{cases} \]

positive constants \( \bar{h}_d \) and \( \sigma_d \) exist such that for all scalars \( h \in (0, \bar{h}_d] \), all index sets \( \alpha \subseteq \{1, \ldots, \ell\} \), and all vectors \( g \in \mathbb{R}^m \), if the system

\[ [(D + CB(h))_{\alpha \bullet}]^T \mu_{\alpha} = g, \quad \mu_{\alpha} \geq 0 \]

has a solution, then it has a solution \( \mu_{\alpha}(g) \) such that \( \|\mu_{\alpha}(g)\| \leq \sigma_d \|g\| \).

- Clearly satisfied with pure control constraints \( (C = 0) \).
- Rather restrictive for pure state constraints \( (D = 0) \).
3 Main Consequences Under (A–E)

• The LQ control problem has an optimal solution in which both the state and costate trajectories are Lipschitz continuous and the control trajectory is integrable;

• such a solution is obtained as the limit of a sequence of numerical trajectories constructed by solving finite-dimensional convex quadratic programs that are guaranteed solvable; and

• a rigorous analysis is provided (for the first time) for the convergence of the solutions to a discrete-time model predictive control approach to a continuous-time optimal solution of the constrained LQ problem.

Side contributions:

• a numerical scheme that unifies time stepping and model predictive control;

• an elementary proof of the necessary and sufficient conditions of optimality and some solution properties.
The Differential Affine Variational Inequality (DAVI)

Find a weak solution \( (x, \lambda, u, \mu) \) with \((x, \lambda)\) absolutely continuous and \((u, \mu)\) integrable such that

\[
x(0) = \xi \text{ and } \lambda(T) = c + Sx(T)
\]

and for almost all \( t \in [0, T] \),

\[
\begin{pmatrix}
\dot{\lambda}(t) \\
\dot{x}(t)
\end{pmatrix} =
\begin{pmatrix}
-p(t) \\
r(t)
\end{pmatrix} +
\begin{bmatrix}
-A^T & -P \\
0 & A
\end{bmatrix}
\begin{pmatrix}
\lambda(t) \\
x(t)
\end{pmatrix} +
\begin{bmatrix}
-Q \\
B
\end{bmatrix} u(t) +
\begin{bmatrix}
C^T \\
0
\end{bmatrix} \mu(t)
\]

\[
0 = q(t) + Q^T x(t) + Ru(t) + B^T \lambda(t) - D^T \mu(t)
\]

\[
0 \leq \mu(t) \perp Cx(t) + Du(t) + f \geq 0
\]

\[
\Rightarrow\quad u(t) \in \arg\min_{u \in U(x(t))} H(x(t), u, \lambda(t))
\]

where

\[
H(x, u, \lambda) \triangleq x^T p + u^T q + \frac{1}{2} x^T Px + x^T Qu + \frac{1}{2} u^T Ru + \lambda^T (Ax + Bu + r)
\]

and

\[
L(x, u, \lambda, \mu) \triangleq H(x, u, \lambda) - \mu^T (Cx + Du + f),
\]

are, respectively, the Hamiltonian and Lagrangian of the LQ problem.
Main Theorem

(Solvability of the DAVI) The DAVI has a weak solution \((x^*, \lambda^*, u^*, \mu^*)\) with both \(x^*\) and \(\lambda^*\) being Lipschitz continuous on \([0, T]\). [Proof by construction].

(Sufficiency of Pontryagin) If \((x^*, \lambda^*, u^*, \mu^*)\) is any weak solution of the DAVI, then the pair \((x^*, u^*)\) is an optimal solution of the LQ problem. [Direct proof].

(Gradient characterization of optimal solutions) If \((\hat{x}, \hat{u})\) and \((\tilde{x}, \tilde{u})\) are any two optimal solutions of the LQ problem, then [Follows from proof of sufficiency]

(a) for almost all \(t \in [0, T]\),

\[
\begin{bmatrix}
P & Q \\
Q^T & R
\end{bmatrix}
\begin{bmatrix}
\hat{x}(t) - \tilde{x}(t) \\
\hat{u}(t) - \tilde{u}(t)
\end{bmatrix} = 0,
\]

(b) \(S\hat{x}(T) = S\tilde{x}(T)\), and

(c) \(c^T(\hat{x}(T) - \tilde{x}(T)) + \int_0^T \begin{bmatrix}
p(t) \\
q(t)
\end{bmatrix}^T \begin{bmatrix}
\hat{x}(t) - \tilde{x}(t) \\
\hat{u}(t) - \tilde{u}(t)
\end{bmatrix} dt = 0.\)

Thus for any optimal solution \((\hat{x}, \hat{u})\) of the LQ problem, a feasible tuple \((\tilde{x}, \tilde{u})\) of the LQ problem is optimal if and only if conditions (a), (b), and (c) hold.
Main Theorem (cont.)

(Necessity of Pontryagin) Let \((x^*, \lambda^*, u^*, \mu^*)\) be the tuple obtained from part (I). A feasible tuple \((\tilde{x}, \tilde{u})\) of the LQ problem is optimal if and only if \((\tilde{x}, \lambda^*, \tilde{u}, \mu^*)\) is a weak solution of the DAVI. [Follows from proofs of previous parts].

(Uniqueness under positive definiteness) If \(R\) is positive definite, then for any two optimal solutions \((\hat{x}, \hat{u})\) and \((\tilde{x}, \tilde{u})\) of the LQ problem, \(\hat{x} = \tilde{x}\) everywhere on \([0, T]\) and \(\hat{u} = \tilde{u}\) almost everywhere on \([0, T]\). [Follows from gradient characterization of optimality].

In this case, the LQ problem has a unique optimal solution \((\hat{x}, \hat{u})\) such that \(\hat{x}\) is continuously differentiable and \(\hat{u}\) is Lipschitz continuous on \([0, T]\), and for any optimal \(\hat{\lambda}\), \(\hat{u}(t) \in \text{argmin}_{u \in U(\hat{x}(t))} H(\hat{x}(t), u, \hat{\lambda}(t))\) for all \(t \in [0, T]\). [Follows from finite-dimensional QP theory].

In conclusion, key is the constructive proof of part (I), the rest follows from direct verification.
The Unified Time-Stepping Algorithm: Overview

- Let $h > 0$ be an arbitrary step size and let $N_h \triangleq T/h - 1$.
- Partition the interval $[0, T]$ into $N_h + 1$ subintervals each of equal length $h$:
  \[ 0 \triangleq t_{h,0} < t_{h,1} < t_{h,2} < \cdots < t_{h,N_h} < t_{h,N_h+1} \triangleq T. \]
- Calculate the iterates $x^h \triangleq \{ x^{h,i} \}_{i=0}^{N_h+1}$ and $u^h \triangleq \{ u^{h,i} \}_{i=1}^{N_h+1}$ by solving $N_h$ finite-dimensional convex quadratic subprograms.
- Construct continuous-time numerical trajectories, $\hat{x}^h$ and $\hat{u}^h$, on the interval $[0, T]$ by piecewise linear and piecewise constant interpolation, respectively; for all $i = 0, \cdots, N_h$:
  \[
  \begin{align*}
  \hat{x}^h(t) & \triangleq x^{h,i} + \frac{t-t_{h,i}}{h} (x^{h,i+1} - x^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}] \\
  \hat{u}^h(t) & \triangleq u^{h,i+1}, \quad \forall t \in (t_{h,i}, t_{h,i+1}].
  \end{align*}
  \]
- Construct auxiliary multiplier trajectories $\hat{\lambda}^h$ and $\hat{\mu}^h$ appropriately.
- Establish the limiting properties of the numerical trajectories $\{ (\hat{x}^h, \hat{u}^h, \hat{\lambda}^h, \hat{\mu}^h) \}$ as $h \downarrow 0$ by proving some key bounds.
The Unified Time-Stepping Algorithm: Setup

Given: a scalar \( \theta \in [0, 1] \), and matrix-valued functions \( E(h) \), \( \hat{E}(h) \), \( A(h) \), and \( B(h) \) satisfying the following conditions:

\[
\lim_{h \downarrow 0} \frac{E(h)}{h} = \lim_{h \downarrow 0} \frac{\hat{E}(h)}{h} = I,
\]

\[
A(h) = I + hA + \sum_{i=2}^{\infty} a_i h^i A^i, \quad \text{and} \quad B(h) \triangleq h B + \sum_{i=1}^{\infty} b_i h^{i+1} A^i B,
\]

for some scalars \( a_i \) and \( b_i \).

- \( \theta = 0 \), \( \hat{E}(h) \triangleq h A(h) \), \( A(h) \triangleq (I - h A)^{-1} \), and \( B(h) \triangleq h A(h) B \)
  \( \Rightarrow \) a standard time stepping method

- \( \theta = 1 \), \( E(h) \triangleq \int_0^h e^{As} ds \), \( A(h) \triangleq e^{A h} \), and \( B(h) \triangleq \int_0^h e^{As} ds B \)
  \( \Rightarrow \) model predictive control.

Let \( p^{h,i} \triangleq p(t_{h,i}) \), \( q^{h,i} \triangleq q(t_{h,i}) \), and \( r^{h,i} \triangleq r(t_{h,i}) \).
The Unified Time-Stepping Algorithm: Exact feasibility

\[
\begin{align*}
\text{minimize} & \quad (x^{h,Nh+1})^T (c + \frac{1}{2} S x^{h,Nh+1}) + \\
&\quad h \sum_{i=0}^{Nh} \left\{ 2 \left( \begin{array}{c}
\theta x^{h,i} + (1 - \theta) x^{h,i+1} \\
u^{h,i+1}
\end{array} \right)^T \left( \begin{array}{c}
p^{h,i+1} \\
q^{h,i+1}
\end{array} \right) + \\
&\quad \left( \begin{array}{c}
\theta x^{h,i} + (1 - \theta) x^{h,i+1} \\
u^{h,i+1}
\end{array} \right)^T \left[ \begin{array}{cc}
P & Q \\
Q^T & R
\end{array} \right] \left( \begin{array}{c}
\theta x^{h,i} + (1 - \theta) x^{h,i+1} \\
u^{h,i+1}
\end{array} \right) \right\} \\
\text{subject to} & \quad x^{h,0} = \xi, \\
&\quad \text{and for } i = 0, 1, \ldots, Nh: \\
\left\{ \\
x^{h,i+1} = \left[ \theta E(h) r^{h,i} + (1 - \theta) \hat{E}(h) r^{h,i+1} \right] + A(h)x^{h,i} + B(h)u^{h,i+1} \\
u^{h,i+1} \in U(x^{h,i+1}) \triangleq \{ u \mid f + C x^{h,i+1} + Du \geq 0 \} \\
\right\}.
\end{align*}
\]

Except for the somewhat unusual discretization of the differential equation: \( \dot{x}(t) = r(t) + Ax(t) + Bu(t) \), the rest is fairly standard.

Major drawback: feasibility, and thus solvability, is not guaranteed!
The Unified Time-Stepping Algorithm: Constraint Relaxation

- Define the minimum constraint residual by solving a linear program:

\[ \rho_h(\xi) \triangleq \min_{\rho; \{x^{h,i}, u^{h,i}\}_{i=1}^{N_h}} \rho \]

subject to \( x^{h,0} = \xi, \quad \rho \geq 0 \)

and for \( i = 0, 1, \ldots, N_h \):

\[
\begin{align*}
\{ x^{h,i+1} &= \left[ \theta E(h) r^{h,i} + (1 - \theta) \hat{E}(h) r^{h,i+1} \right] + A(h)x^{h,i} + B(h)u^{h,i+1} \\
& \quad + Cx^{h,i+1} + Du^{h,i+1} + f + \rho 1 \geq 0 \}
\end{align*}
\]

- Key property: \( \lim_{h \downarrow 0} \rho_h(\xi) = 0. \) [Consequence of the feasibility assumption.]
The Relaxed $\tilde{\text{QP}}^h$ (guaranteed feasible, thus solvable)

\begin{align*}
\text{minimize} & \quad (x^{h,N_h+1})^T (c + \frac{1}{2} S x^{h,N_h+1}) + \\
& \quad \frac{h}{2} \sum_{i=0}^{N_h} \left\{ 2 \begin{pmatrix} \theta x^{h,i} + (1 - \theta) x^{h,i+1} \\ u^{h,i+1} \end{pmatrix}^T \begin{pmatrix} p^{h,i+1} \\ q^{h,i+1} \end{pmatrix} + \begin{pmatrix} \theta x^{h,i} + (1 - \theta) x^{h,i+1} \\ u^{h,i+1} \end{pmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{pmatrix} \theta x^{h,i} + (1 - \theta) x^{h,i+1} \\ u^{h,i+1} \end{pmatrix} \right\} \\
\text{subject to} & \quad x^{h,0} = \xi \quad \text{and for } i = 0, 1, \ldots, N_h:\n& \quad x^{h,i+1} = \begin{pmatrix} \theta E(h) r^{h,i} + (1 - \theta) \hat{E}(h) r^{h,i+1} \\ A(h) x^{h,i} + B(h) u^{h,i+1} \end{pmatrix} + \begin{pmatrix} \lambda^{h,i} \\ \mu^{h,i+1} \end{pmatrix} \\
& \quad f + C x^{h,i+1} + D u^{h,i+1} + \rho_h(\xi) > 0
\end{align*}

If a Lipschitz feasible control with a known constant is available, an alternative relaxed QP that is guaranteed feasible can be defined, avoiding the solution of the minimum-residual linear program.
Main Convergence

Define the $\lambda$-trajectory and $\mu$-trajectory similarly to the $x$- and $u$-trajectory, respectively; namely, for $i = 0, \cdots, N_{h}$, with $\lambda^{h,N_{h}+1} \triangleq c + Sx^{h,N_{h}+1},$

\[
\hat{\lambda}^{h}(t) \triangleq \lambda^{h,i} + \frac{t-t_{h,i}}{h} (\lambda^{h,i+1} - \lambda^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}]
\]

\[
\hat{\mu}^{h}(t) \triangleq h^{-1}\mu^{h,i+1}, \quad \forall t \in (t_{h,i}, t_{h,i+1}]
\]

- $\exists$ a sequence of step sizes $\{h_{\nu}\} \downarrow 0$ such that the two limits exist on $[0, T]$:

\[
(\hat{x}^{h_{\nu}}, \hat{\lambda}^{h_{\nu}}) \to (\hat{x}, \hat{\lambda}) \text{ uniformly and } (\hat{u}^{h_{\nu}}, \hat{\mu}^{h_{\nu}}) \to (\hat{u}, \hat{\mu}) \text{ weakly in } L^{2};
\]

moreover, $\hat{x}$ and $\hat{\lambda}$ are Lipschitz continuous.

- The sequences $\left\{ \begin{bmatrix} P & Q \\ QT & R \end{bmatrix} \left( \begin{bmatrix} \hat{x}^{h} \\ \hat{u}^{h} \end{bmatrix} \right) \right\}$ and $\left\{ DT\hat{\mu}^{h} \right\}$ converge, respectively, to $\left[ \begin{bmatrix} P & Q \\ QT & R \end{bmatrix} \left( \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} \right) \right]$ and $DT\hat{\mu}$ uniformly on $[0, T]$.

- Any limit tuple $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ from (a) is a weak solution of the DAVI; thus $(\hat{x}, \hat{u})$ is an optimal solution of the LQ problem.

\[\blacksquare\]
Key bounds in the proof

- Positive scalars $\bar{h}$, $\eta$, $\Psi_u$, and $L$ exist such that for all $h \in (0, \bar{h}]$, KKT multipliers $(\lambda^h, \mu^h)$ exist such that for all optimal solutions $(x^h, u^h)$ of the $(\hat{Q}P^h)$ and for all $i = 0, \cdots, N_h$,

$$\max \left( \begin{array}{c}
\|x^{h,i+1}\|, \|u^{h,i+1}\|, \|\lambda^{h,i}\|, h^{-1} \|\mu^{h,i+1}\| \\
\text{state}, \text{control}, \text{co-state}, \text{multiplier}
\end{array} \right) \leq \eta (1 + \Psi_u),$$

and for all $i = 0, \cdots, N_h - 1$,

$$\max \left\{ \left\| \begin{bmatrix}
Q \\
R
\end{bmatrix} (u^{h,i+2} - u^{h,i+1}) \right\|, h^{-1} \| DT(\mu^{h,i+2} - \mu^{h,i+1}) \| \right\}$$

$$\leq L \left[ \| q^{h,i+2} - q^{h,i+1} \| + \| x^{h,i+2} - x^{h,i+1} \| + \| x^{h,i+1} - x^{h,i} \| + \| \lambda^{h,i+1} - \lambda^{h,i} \| \right].$$

**Remark:** Bounding $h^{-1} \|\mu^{h,i+1}\|$ requires the perturbed Hoffman bound that in turn depends on condition (E).
• It then follows that
\[ h^{-1} \left\{ \| x^{h,i+1} - x^{h,i} \|, \| \lambda^{h,i+1} - \lambda^{h,i} \| \right\}_{i=0}^{N_h} \]
and
\[ h^{-1} \left\{ \left\| \begin{bmatrix} \frac{Q}{R} \\ 0 \end{bmatrix} (u^{h,i+2} - u^{h,i+1}) \right\|, h^{-1} \| DT(\mu^{h,i+2} - \mu^{h,i+2}) \| \right\}_{i=0}^{N_h-1} \]
are all bounded uniformly for all \( h > 0 \) sufficiently small.

• Applying the \textbf{Arzela-Ascoli theorem} in functional analysis completes the proof. \qed
Lecture IV

On differential linear-quadratic Nash games with mixed state-control constraints

Tuesday September 11, 2012, 4:30–5:30 PM

This is joint work with Dane Schiro, a graduate student at the University of Illinois at Urbana-Champaign.

Research extends the static Nash games to a continuous-time setting.
Contents of Presentation

- Definition and assumptions of the differential Nash game

- Recap: of Lecture III

- The symmetric case
  - equivalent to a single optimal control problem

- The asymmetric case
  - convergence of best-response iterations
The open-loop differential Nash game

This is an $\mathcal{F}$-person non-cooperative game in the finite time horizon $[0, T]$.

Each player $i = 1, \cdots, \mathcal{F}$ chooses an absolutely continuous state function $x_i : [0, T] \to \mathbb{R}^{n_i}$ and a bounded measurable (thus integrable) control function $u_i : [0, T] \to \mathbb{R}^{m_i}$ for some positive integers $n_i$ and $m_i$ to solve a linear-quadratic (LQ) optimal control problem parameterized the rival’s variables $(x_{-i}, u_{-i})$:

$$\min_{x, u} \theta_i(x, x_{-i}, u, u_{-i}) \triangleq x(T)^T \left[ c_i + \sum_{i' = 1}^{\mathcal{F}} W_{ii'} x_{i'}(T) \right] +$$

$$\int_0^T \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix}^T \begin{pmatrix} p_i(t) \\ q_i(t) \end{pmatrix} + \sum_{i' = 1}^{\mathcal{F}} \begin{pmatrix} P_{ii'} & Q_{ii'} \\ R_{ii'} & S_{ii'} \end{pmatrix} \begin{pmatrix} x_{i'}(t) \\ u_{i'}(t) \end{pmatrix} \right] dt$$

subject to

- **dynamics** $\dot{x}_i(t) = r_i + A_{ii} x_i(t) + B_{ii} u_i(t)$ for almost all $t$
- **algebraic constraint** $f_i + C_{ii} x_i(t) + D_{ii} u_i(t) \geq 0$ for almost all $t$
- **initial condition** $x_i(0) = \xi_i$

Let $U_i(x_i) \triangleq \{ u_i \mid f_i + C_{ii} x_i + D_{ii} u_i \geq 0 \}$ be the state-dependent set of feasible controls.
An aggregated pair \((\mathbf{x}^*, \mathbf{u}^*)\), where \(\mathbf{x}^* \triangleq (x^*_i)_{i \in \mathcal{F}}\) and \(\mathbf{u}^* \triangleq (u^*_i)_{i \in \mathcal{F}}\) is a Nash equilibrium (NE) of the above game if for each \(i = 1, \cdots, \mathcal{F}\),

\[
(x^*_i, u^*_i) \in \text{argmin}_{(x_i, u_i)} \theta_i(x_i, x^*_{-i}, u_i, u^*_{-i})
\]

subject to \((x_i, u_i)\) feasible to (1).

Interested in existence, uniqueness, regularity, and computation of a solution.

**Assumptions**

\(A_i\) the matrices \(W_{ii}\) and \(\Xi_{ii}\) are symmetric positive semidefinite (as opposed to positive definiteness as assumed in much of the control literature);

\(B_i\) the functions \(p_i(t)\) and \(q_i(t)\) are Lipschitz continuous on \([0, T]\);

\(C_i\) a continuously differentiable function \(\hat{x}^f_{fs}\) with \(\hat{x}^f_{fs}(0) = \xi_i\) and a continuous function \(\hat{u}^f_{fs}\) exist such that for all \(t \in [0, T]\): \(\hat{u}^f_{fs}(t) \in U_i(\hat{x}^f_{fs}(t))\) and \(d\hat{x}^f_{fs}(t)/dt = r_i + A_i\hat{x}^f_{fs}(t) + B_i\hat{u}^f_{fs}(t)\);

\(D_i\) \([S_{ii}u_i = 0, D_iu_i \geq 0]\) implies \(u_i = 0\) (a primal weak coercivity condition; going along with \((A_i)\));

\(E_i\) \([D^T_i \mu_i = 0, \mu_i \geq 0]\) implies \((C_iA^j_iB_i)^T \mu_i = 0\) for all integers \(j = 0, \cdots, n_i - 1\), or equivalently, for all nonnegative integers \(j\) (a dual condition).
The (boundary-value) differential affine variational inequality (DAVI) associated with the LQ problem \([1]\) is:

\[
\begin{pmatrix}
\dot{\lambda}_i(t) \\
\dot{x}_i(t)
\end{pmatrix} = 
\begin{pmatrix}
-p_i(t) - \sum_{i' \neq i} [P_{ii'}x_i'(t) + Q_{ii'}u_i'(t)] \\
0
\end{pmatrix} + 
\begin{bmatrix}
-A_i^T & -2P_{ii} \\
0 & A_i
\end{bmatrix}
\begin{pmatrix}
\lambda_i(t) \\
x_i(t)
\end{pmatrix} + 
\begin{bmatrix}
-(Q_{ii} + R_{ii}^T)
\\
B_i
\end{bmatrix} u_i(t) + 
\begin{bmatrix}
C_i^T \\
0
\end{bmatrix} \mu_i(t)
\]

\[
\begin{cases}
0 = q_i(t) + \sum_{i' \neq i} [R_{ii'}x_{i'}(t) + S_{ii'}u_{i'}(t)] + (Q_{ii}^T + R_{ii}) x_i(t) \\
+2 S_{ii}u_i(t) + B_i^T \lambda_i(t) - D_i^T \mu_i(t) \\
0 \leq \mu_i(t) - f_i + C_i x_i(t) + D_i u_i(t) \geq 0
\end{cases}
\]

these conditions imply that \(u_i(t) \in \arg\min_{u_i \in U_i(x_i(t))} H_i(x(t), u_i, u_{-i}(t), \lambda_i(t))\) Hamiltonian

\[
x_i(0) = \xi_i \text{ and } \lambda_i(T) = c_i + 2W_{ii}x_i(T) + \sum_{i' \neq i} W_{ii'}x_{i'}(T)
\]

ODE + variational conditions + initial/boundary conditions
Properties under (A)–(E)

- **(Solvability of the DAVI)** Provided that the pair \((x_{-i}, u_{-i})\) is Lipschitz continuous, the DAVI has a weak solution \((x_i^*, u_i^*, \lambda_i^*, \mu_i^*)\). [Proof by a constructive time-stepping scheme.]

- **(Sufficiency of Pontryagin)** If \((x_i^*, u_i^*, \lambda_i^*, \mu_i^*)\) is any weak solution of the DAVI, then the pair \((x_i^*, u_i^*)\) is an optimal solution of the problem (1).

- **(Necessity of Pontryagin)** Let \((x_i^*, u_i^*, \lambda_i^*, \mu_i^*)\) be the tuple obtained from part (I). A feasible tuple \((\tilde{x}_i, \tilde{u}_i)\) of (1) is optimal if and only if \((\tilde{x}_i, \tilde{u}_i, \lambda_i^*, \mu_i^*)\) is a weak solution of the DAVI.

- **Variational characterization of optimality** The optimal solution \((x_i^*, u_i^*)\) is characterized by a variational principle.

- **(Uniqueness under positive definiteness)** If \(S_{ii}\) is positive definite and the pair \((x_{-i}, u_{-i})\) is Lipschitz continuous, then (1) has a unique optimal solution \((\hat{x}_i, \hat{u}_i)\) such that \(\hat{x}_i\) is continuously differentiable and \(\hat{u}_i\) is Lipschitz continuous on \([0, T]\).

□
The Symmetric Case

Consider the aggregated LQ optimal control problem in the variables \((x, u)\):

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{\mathcal{F}} x_i(T)^T \left[ c_i + \sum_{i'=1}^{\mathcal{F}} W_{ii'} x_i'(T) \right] + \left( x_i(t) \right)^T \left[ \begin{array}{cc} P_{ii} & Q_{ii} \\ R_{ii} & S_{ii} \end{array} \right] \left( u_i(t) \right) \\
& \quad + \int_0^T \sum_{i=1}^{\mathcal{F}} \left\{ \left( \begin{array}{c} x_i(t) \\ u_i(t) \end{array} \right)^T \left( \begin{array}{c} p_i(t) \\ q_i(t) \end{array} \right) + \frac{1}{2} \sum_{i' \neq i}^{\mathcal{F}} \left[ \begin{array}{cc} P_{ii'} & Q_{ii'} \\ R_{ii'} & S_{ii'} \end{array} \right] \left( \begin{array}{c} x_i'(t) \\ u_i'(t) \end{array} \right) \right\} dt
\end{align*}
\]

subject to

- for all \(i \in \{1, \cdots, \mathcal{F}\}\), (aggregated constraints),
- \text{dynamics} \quad \dot{x}_i(t) = r_i + A_i x_i(t) + B_i u_i(t) \quad \text{for almost all } t
- \text{algebraic constraint} \quad f_i + C_i x_i(t) + D_i u_i(t) \geq 0 \quad \text{for almost all } t
- \text{initial condition} \quad x_i(0) = \xi_i.

Let \(\Xi_{ii'} \triangleq \begin{cases} 
2 \left[ \begin{array}{cc} P_{ii} & 1/2 \left( Q_{ii} + R_{ii}^T \right) \\ 1/2 \left( Q_{ii}^T + R_{ii} \right) & S_{ii} \end{array} \right] & \text{if } i = i' \\
\left[ P_{ii'} Q_{ii'} \\ R_{ii'} S_{ii'} \right] & \text{if } i \neq i'.
\end{cases}\)
In general, $\Xi_{ii'} \neq \Xi_{i'i}$ for $i \neq i'$, reflecting the asymmetric impact of the strategy of rival $i'$ on player $i$'s objective and vice versa.

Theorem. Suppose that the matrices $W \triangleq [W_{ii'}]_{i,i'=1}^{n}$ and $\Xi \triangleq [\Xi_{ii'}]_{i,i'=1}^{n}$ are symmetric positive semidefinite.

- **(Equivalence)** A pair $(x^*, u^*)$ is a NE if and only if $(x^*, u^*)$ is an optimal solution of the aggregated LQ optimal control problem.

- **(Existence)** If conditions $(B_i), (C_i), \text{ and } (E_i)$ hold for all $i$, and

\[
(D) \quad \left[ \sum_{i'=1}^{\mathcal{F}} S_{ii'} u_{i'} = 0 \text{ and } D_i u_i \geq 0 \text{ for all } i \in \{1, \ldots, \mathcal{F}\} \right],
\]

implies $[u_i = 0 \text{ for all } i \in \{1, \ldots, \mathcal{F}\}]$

then a NE exists such that $x^*$ is absolutely continuous and $u^*$ is square-integrable on $[0, T]$;

- **(Uniqueness)** If in addition, the matrix $S \triangleq [S_{ii'}]_{i,i'=1}^{n}$ is positive definite, then $(x^*, u^*)$ is the unique NE such that $x^*$ is continuously differentiable and $u^*$ is continuous on $[0, T].$
The Asymmetric Case

(A) For all $i = 1, \cdots, \mathcal{F}$, the matrices $\Xi_{ii}$, are positive definite with minimum eigenvalues $\sigma_{\Xi} > 0$; the matrices $W_{ii}$ remains (symmetric) positive semidefinite; [pertaining to individual players' objectives]

(W) For all $i = 1, \cdots, \mathcal{F}$, the matrices $W_{ii} = 0$ for all $i' \neq i$.

(D) For all $i = 1, \cdots, \mathcal{F}$, the following implication holds: $D_{ii}u_{i} \geq 0 \Rightarrow u_{i} = 0$.

(E) For all $i = 1, \cdots, \mathcal{F}$, the Markov parameters $C_{i}A_{i}^{j}B_{i} = 0$ for all $j = 0, \cdots, n_{i} - 1$.

Define the matrix $\Gamma \triangleq [\Gamma_{ii}]_{i,i'=1}^{\mathcal{F}}$, where

$$
\Gamma_{ii'} \triangleq \begin{cases} 
0 & \text{if } i = i' \\
\frac{1}{\sqrt{\sigma_{\Xi}^{i} \sigma_{\Xi}^{i'}}} \| \Xi_{ii'} \|, & \text{if } i \neq i'.
\end{cases}
$$
The best-response algorithm

Given a pair of state-control trajectories \((x^{\nu}, u^{\nu})\) at the beginning of iteration \(\nu + 1\), where \(x^{\nu}\) is continuously differentiable and \(u^{\nu}\) is Lipschitz continuous, we compute the next pair of such trajectory \((x^{\nu+1}, u^{\nu+1})\) by solving \(\mathcal{F}\) LQ optimal control problem (1), where for \(i = 1, \ldots, \mathcal{F}\), the \(i\)-th such LQ problem solves for the pair \((x^{\nu+1}_i, u^{\nu+1}_i)\) from (1) by fixing \((x_j, u_j)\) at \((x^{\nu}_j, u^{\nu}_j)\) for all \(j \neq i\), i.e.,

\[
(x^{\nu+1}_i, u^{\nu+1}_i) \in \text{argmin}_{(x_i, u_i)} \theta_i(x_i, x^{\nu}_{-i}, u_i, u^{\nu}_{-i})
\]

subject to \((x_i, u_i)\) feasible to (1).

This is the Jacobi version of the method.
Theorem. Under assumptions \((\hat{A}), (B), (W), (\hat{D}), (\hat{E})\), the following statements hold for the sequence \(\{(x^\nu_i, u^\nu_i)\}\) generated by the best-response algorithm.

(Well-definedness) The sequence \(\{(x^\nu_i, u^\nu_i)\}\) is well-defined with \(x^\nu_i\) being continuously differentiable and \(u^\nu_i\) Lipschitz continuous on \([0, T]\) for all \(\nu\).

(Contraction and strong convergence) If \(\rho(\Gamma) < 1\), then the sequence \(\{(x^\nu_i, u^\nu_i)\}\) converges strongly to a pair \((x^\infty_i, u^\infty_i)\) that is the unique NE of the differential LQ game.