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Brief Paper

Practical Stabilization of a Class of Nonlinear Systems with Partially Known Uncertainties*

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Key Words—Uncertain systems; unknown uncertainties; practical stabilization.

Abstract—In this paper we deal with robust control of a class of nonlinear systems which contain uncertainties. It can be viewed as an extension of the work in Corless and Leitmann [*IEEE Trans. Autom. Control*, **AC-26**, 1139–1144 (1981)] for the cases where the vector of uncertainties is only partially known. To cope with the uncertainties, an adaptive controller using a dead-zone and a hysteresis function is proposed, and both the uniform boundedness of all the closed-loop signals and uniform ultimate boundedness of the system state are guaranteed. In contrast with some previous attempts to relax the a priori knowledge on the uncertainties bounds by using a discontinuous control law, we propose continuous control laws in this paper. Hence, chattering problems (which have practical importance) can be avoided.

1. *Introduction*

Control algorithms, using techniques based on Lyapunov's direct method for uncertain systems, have been studied since the beginning of the 1960s (Johnson, 1964; Grayson, 1963; Monopoli, 1965, 1966). During the last fifteen years, numerous papers dealing with control of continuous-time systems containing uncertainties have been published. (We refer the reader to Corless (1993), Leitmann (1993) and Zinober (1990) for overviews, and the numerous references therein.) The robust controllers studied in those references generally hinge on three main assumptions: (i) the system state vector is available for measurement; (ii) the so-called matching conditions (which characterize the way the uncertainties enter into state equation) are verified; and (iii) the uncertain unknown elements are assumed to belong to a known compact set, i.e. an upper bound (possibly time varying and state dependent) of the uncertainties vector norm is supposed to be known. Given these assumptions, it is shown that there exists a class of continuous-time controllers that insure the convergence of the state in an arbitrarily small neighbourhood of the origin in finite time. In particular, the saturation nonlinearity, which consists of a sign function enveloped around zero by a boundary layer, is widely used.

Recently, several authors have proposed new control laws which allow the relaxation of assumption (ii) without any restriction on the size of the uncertainties, thus improving the works of Barmish and Leitmann (1982), Chen and Leitmann (1987) and Ryan and Corless (1984). Freeman and Kokotovic (1992) apply the so-called backstepping idea to a particular class of nonlinear systems with unmatched uncertainties. Marino and Tomei (1992) also employ backstepping ideas to study the case where the uncertainties vanish at zero. It is noteworthy that the backstepping method can also be used to relax assumption (i) for certain classes of nonlinear systems (Marino and Tomei, 1992). Qu *et al.* (1991) obtain local results for flexibility joint manipulators using a different method.

A salient feature of those schemes is that the state feedback explicitly depends on the upper bound of the uncertainties. However, in most cases the upper bound is a linear function of some parameters which, therefore, have to be known. Some attempts to relax assumption (iii) have been proposed in the literature (Liao *et al.*, 1990; Corless and Leitmann, 1983; Singh, 1985; Fu, 1992; Dawson *et al.*, 1990, 1992; Leung *et al.*, 1991; Chen, 1992).

In an interesting paper Liao *et al.* (1990) have used variable structure control to make robust uncertain state feedback input–output linearizable nonlinear systems. A continuous-time input is used, and it is proved that all the closed-loop signals are bounded. Corless and Leitmann (1983) have proposed a class of controllers which aims to generalize their work in Corless and Leitmann (1981). However, the scheme proposed in Corless and Leitmann (1983) is globally stable under the restrictive condition that the boundary layer in the saturation nonlinear input exponentially converges toward zero, i.e. the input will generally become discontinuous in practice. Hence, it may lead to chattering phenomena. Similar methods are proposed in Singh (1985), Yoo and Chung (1992), Fu (1992) and Dawson *et al.* (1990, 1992). In Fu (1992) a solution to the problem of chattering is proposed, based on the method in Corless and Leitmann (1983). A simple σ-modification (Ioannou and Kokotovic, 1984) is applied to the upperbound-parameter estimates while the boundary layer in the saturation function is kept strictly positive. (The undesirable chattering problems can be avoided by choosing a saturation input with a large enough boundary layer, see e.g. Spong and Vidyasagar (1989).) However, in this case the closed-loop system state converges within a set whose size depends on the unknown upperbound parameters. Furthermore, it is known in parameter adaptive theory (Hsu and Costa, 1987) that such modifications can change the qualitiative stability behaviour of the closed-loop system in the presence of certain types of disturbances. Chen (1992) proposes a scheme that guarantees convergence of the state vector in the neighborhood (of non-arbitrary size) of the origin, when the uncertainties are cone bounded and the measured state vector contains some noise. (The nominal system is assumed to be linear in Chen (1992).)

This paper aims at extending the work in Corless and Leitmann (1981), when the uncertainties upper bounds are partially known, i.e. they are linear in some unknown constant parameters. It is organized as follows. In Section 2, we present the class of systems to be studied and the basic assumptions. The main result is presented in Section 3. Conclusions are finally given in Section 4. Stability proofs and definitions follow in Appendices A and B.

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2. *System and assumptions*

The following class of nonlinear systems, as used in Corless and Leitmann (1981), will be considered in this paper:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t) + B(\mathbf{x}(t), t)(u(t) + e(\mathbf{x}(t), t)), \qquad (1)$$

where $\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t) \in R^n$ is the available state vector, $u(t) \in R^m$ is the control input and $e(\mathbf{x}, t)$ is the uncertainty. Let us also introduce the following assumptions:

*A*1. There exists a known function $\rho(\mathbf{x}, t): R^n \times R \to R^p$ and an unknown constant vector $\alpha^* \in R^p$ such that for all $\mathbf{x} \in R^n$ and all $t > 0$ we have

$$|e(\mathbf{x}, t)| \le \rho^{\mathrm{T}}(\mathbf{x}, t)\alpha^*, \qquad (2)$$

with $\rho_i(\mathbf{x}, t) > 0$ for all $\mathbf{x}$ such that $|\mathbf{x}| > 0$, $i = 1, 2, \ldots, p$.

*A*2. $f(.,.)$, $B(.,.)$, $e(.,.)$ and $\rho(.,.)$ verify the Caratheodory conditions, i.e. (see e.g. Filipov (1988)) for all t and $\mathbf{x}$ in a bounded domain $D = I \times B$ of the $(t, \mathbf{x})$-space: (i) they are continuous in $\mathbf{x}$ for almost all t; (ii) they are Lebesgue measurable in t for each $\mathbf{x}$; and (iii) there exist Lebesgue summable functions $m_i^B(t)$ $1 \le i \le 4$ such that on D

$$|f(\mathbf{x}, t)| \le m_1^B(t), \quad |B(\mathbf{x}, t)| \le m_2^B(t), \quad |e(\mathbf{x}, t)| \le m_3^B(t) \quad \text{and } |\rho(\mathbf{x}, t)| \le m_4^B(t).$$

Therefore, (1) can be considered as a Caratheodory equation provided u is defined as a Caratheodory function of $\mathbf{x}$ and t. Furthermore, we assume that each one of these functions is locally Lipschitz continuous in its first argument $\mathbf{x}$.† Thus, under these basic assumptions the system in (1) is well-posed (Sontag, 1990), in the sense that local existence and uniqueness of solutions can be proved.

*A*3. The equilibrium point $\mathbf{x} = 0$ of the uncontrolled nominal system $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ is globally uniformly asymptotically stable (GUAS) and there exists a locally Lipschitz Lyapunov function‡ $V_{\mathbf{x}}(\mathbf{x}, t)$ such that:

(i) $\gamma_1(|\mathbf{x}|) \le V_{\mathbf{x}}(\mathbf{x}, t) \le \gamma_2(|\mathbf{x}|)$; (3)

(ii) $\dot{V}_{\mathbf{x}}(\mathbf{x}, t) = \dfrac{\partial V_{\mathbf{x}}}{\partial t} + \left(\dfrac{\partial V_{\mathbf{x}}}{\partial \mathbf{x}}\right)^{\mathrm{T}} f(\mathbf{x}, t) \le -\gamma_3(|\mathbf{x}|)$; (4)

(iii) $V_{\mathbf{x}}$ is proper, i.e. its level sets are compact subsets of the state space;

where all functions denoted as $\gamma_i: R^+ \to R^+$ are known class K functions.§

3. *The adaptive robust controller*

In Corless and Leitmann (1983) the authors have presented an extension of their work in Corless and Leitmann (1981) to the case when α^* is unknown. However, stability and convergence results are obtained in these references under the restrictive condition that the boundary layer size of the saturation function verifies

$$\dot{\varepsilon}(t) = -b\varepsilon(t), \quad b > 0, \qquad (5)$$

so that ε exponentially converges to zero. We will proceed as in Corless and Leitmann (1983), i.e. we will replace the control input α^* in (2) by a time-varying 'estimate' $\hat{\alpha}(t)$. Then, the problem of concern is how to choose the update-law for $\hat{\alpha}$ such that all the signals in the closed-loop system remain bounded (in particular $\hat{\alpha}$ itself), and $\mathbf{x}(t)$ is ultimately bounded with respect to a compact set of size ε. Assume that u in (1) is given by

$$u = \begin{cases} -\dfrac{g^{\mathrm{T}}(\mathbf{x}, t)}{|g(\mathbf{x}, t)|}\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} & \text{if } |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} > \varepsilon \\ -\dfrac{g^{\mathrm{T}}(\mathbf{x}, t)}{\varepsilon}(\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha})^2 & \text{if } |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} \le \varepsilon, \end{cases} \qquad (6)$$

† A function is locally Lipschitz continuous if for each compact subset Ω of its domain of definition there exists a constant L depending only on Ω, for which $|f(x) - f(y)| \le L\,|x - y|$ for all x, y in Ω.

‡ Existence of such a $V_{\mathbf{x}}$ satisfying (i) and (ii) is guaranteed by Massera's theorem (Massera, 1956).

§ $\gamma_i(\cdot)$ is strictly increasing, $\gamma_i(0) = 0$. The inverse functions $\gamma_i^{-1}: R^+ \to R^+$ are, therefore, well defined.

where $\hat{\alpha}$ will be defined later. Let us consider the following positive definite function

$$V(\mathbf{x}, \tilde{\alpha}, t) = V_{\mathbf{x}}(\mathbf{x}, t) + \tfrac{1}{2}\tilde{\alpha}^{\mathrm{T}}\tilde{\alpha}, \qquad (7)$$

where $\tilde{\alpha} = \hat{\alpha} - \alpha^*$. Taking the derivative of V along the trajectories of the system (1) leads to

$$\dot{V}(\mathbf{x}, \tilde{\alpha}, t) = \frac{\partial V_{\mathbf{x}}}{\partial t} + \left(\frac{\partial V_{\mathbf{x}}}{\partial \mathbf{x}}\right)^{\mathrm{T}} f(\mathbf{x}, t) + \left(\frac{\partial V_{\mathbf{x}}}{\partial \mathbf{x}}\right)^{\mathrm{T}} B(\mathbf{x}, t)\{u + e(\mathbf{x}, t)\} + \tilde{\alpha}^{\mathrm{T}}\dot{\hat{\alpha}} \qquad (8)$$

and with (4) we get

$$\dot{V}(\mathbf{x}, \tilde{\alpha}, t) \le -\gamma_3(|\mathbf{x}|) + g(\mathbf{x}, t)u + |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\alpha^* + \tilde{\alpha}^{\mathrm{T}}\dot{\hat{\alpha}}, \qquad (9)$$

where

$$g(\mathbf{x}, t) = \left(\frac{\partial V_{\mathbf{x}}}{\partial \mathbf{x}}\right)^{\mathrm{T}} B(\mathbf{x}, t).$$

Then, introducing (6) into (9) we obtain the following inequalities:

(i) if $|g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} > \varepsilon$

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) - |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\tilde{\alpha} + \tilde{\alpha}^{\mathrm{T}}\dot{\hat{\alpha}}; \qquad (10)$$

and

(ii) if $|g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} \le \varepsilon$

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) - \frac{|g(\mathbf{x}, t)|^2}{\varepsilon}(\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha})^2 + |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\alpha^* + \tilde{\alpha}^{\mathrm{T}}\dot{\hat{\alpha}}. \qquad (11)$$

Assume now that $\hat{\alpha}$ is updated as follows:

$$\dot{\hat{\alpha}} = \sigma(|\mathbf{x}|)\,|g(\mathbf{x}, t)|\,\rho(\mathbf{x}, t), \quad \hat{\alpha}(t_0) \ge 0, \qquad (12)$$

where σ is depicted in Fig. 1, with

$$R_1 = \gamma_3^{-1}(2\varepsilon + h); \quad R_2 = \gamma_1^{-1} o \gamma_2 o \gamma_3^{-1}(2\varepsilon + h); \quad h > 0, \text{ arbitrary}. \qquad (13)$$

Before going on with the stability analysis, note that the following relationships are verified.‖ Given any $\eta \in R^+$, $t \ge t_0$, $\tilde{\alpha} \in R$ we have

$$V^{-1}[\eta] \subseteq V_{\mathbf{x}}^{-1}[\eta] \subseteq \gamma_1^{-1}[\eta]. \qquad (14)$$

First, notice, that if $\hat{\alpha}$ was constant with $\hat{\alpha} \ge \alpha^*$, then $\mathbf{x}$ would be GUUB (see definition in Appendix A) with respect to the ball B_{R_2}. (This can be shown using the arguments in Corless and Leitmann (1981); see also Khalil (1992).) Let us denote the different situations as follows:

$$s_1: |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} > \varepsilon; \quad s_2: |g(\mathbf{x}, t)|\,\rho^{\mathrm{T}}(\mathbf{x}, t)\hat{\alpha} \le \varepsilon;$$
$$s_3: \sigma = 1; \quad s_4: \sigma = 0.$$

The following combinations can occur a priori: s_1 and s_3; s_1 and s_4; s_2 and s_3; and s_2 and s_4. Indeed nothing guarantees that s_1 implies that $|\mathbf{x}|$ will be smaller or greater than, say, $\gamma_3^{-1}(2\varepsilon + h)$. Let us analyse in more detail the value taken by $\dot{V}(\mathbf{x}, \tilde{\alpha}, t)$ in each case:

‖ We introduce the following notations: Let γ_1 and γ_2 be class K functions. Then, given any $\varepsilon > 0$, $\gamma_1^{-1} o \gamma_2(\varepsilon)$ is the value of the function $\gamma_1^{-1} o \gamma_2(\cdot)$ at ε, while $\gamma_1^{-1}[\gamma_2(\varepsilon)]$ denotes the compact set S defined as: $S = \{\mathbf{x} \in R^n / \gamma_1(|\mathbf{x}|) \le \gamma_2(\varepsilon)\}$.

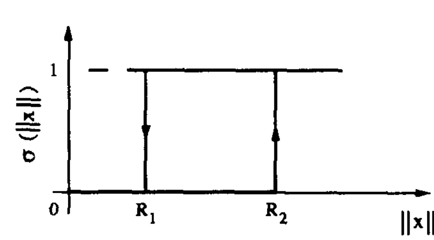

Fig. 1. The adaptation modulation function.

*C*1. s_1–s_3.
We get

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) \le -2\varepsilon < 0. \quad (15)$$

Thus $\dot{V}$ is clearly strictly negative for all $\varepsilon > 0$, $\hat{\alpha}$ is strictly increasing.
*C*2. s_1–s_4.
We obtain

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) - |g(\mathbf{x}, t)|\, \rho^T(\mathbf{x}, t)\tilde{\alpha}. \quad (16)$$

$\hat{\alpha}$ is frozen. If $\tilde{\alpha} > 0$, then $\dot{V} \le -\gamma_3(|\mathbf{x}|)$ otherwise $\dot{V}(\mathbf{x}, \tilde{\alpha}, t)$ may take positive values. However, if $\mathbf{x}$ grows then we are back to the previous case. Also, by construction $|g(\mathbf{x}, t)|\, \rho^T(\mathbf{x}, t)\tilde{\alpha}$ is bounded as $\mathbf{x}$ is bounded.
*C*3. s_2–s_3.
From (11), (12) and (13) it follows that:

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) + \varepsilon \le -\varepsilon - h < 0. \quad (17)$$

$\dot{V}$ is strictly negative outside the ball B_{R_1}, and $\hat{\alpha}$ is non-decreasing.
*C*4. s_2–s_4.
From (11) we get

$$\dot{V} \le -\gamma_3(|\mathbf{x}|) + |g(\mathbf{x}, t)|\, \rho^T(\mathbf{x}, t)\alpha^*. \quad (18)$$

$\hat{\alpha}$ is frozen. Note, that as soon as $\hat{\alpha} \ge \alpha^*$, then using (11) we retrieve the same inequality as in (17). We now have the following lemma:

Lemma. (i) Given any bounded initial condition in (1) the closed-loop system (1), (6), (12) has a unique solution $\bar{\mathbf{x}}^T = (\mathbf{x}^T(t), \hat{\alpha}^T(t))$ on $[t_0, +\infty)$ and this solution is uniformly bounded.

(ii) The system state $\mathbf{x}(t)$ is GUUB in the sense that given any $\varepsilon > 0$ and $h > 0$ in (6), the total time spent by $\mathbf{x}(t)$ outside the set $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ is finite. Moreover, the upper bound estimate $\hat{\alpha}$ converges to a constant finite value $\hat{\alpha}_f$. ■

Proof. The proof of the lemma is in Appendix B.

Remarks. (i) The adaptive robust scheme will generally require much less on-line calculation than the classical adaptive control laws. For example, in the case of adaptive robust control of rigid manipulators, one can choose α^* as a two-dimensional vector (Dawson *et al.*, 1992; Yoo and Chung, 1992) independently of the number of degrees of freedom. Note, moreover, that one may define in equation (2)

$$\rho^T(\mathbf{x}, t)\alpha^* \le |\rho(\mathbf{x}, t)|\, |\alpha^*| = \bar{\rho}\bar{\alpha}^*$$

or

$$\rho^T(\mathbf{x}, t)\alpha^* \le (\rho_1 + \cdots + \rho_p) \max(\alpha_1^*, \ldots, \alpha_p^*) = \bar{\rho}\bar{\alpha}^*.$$

Thus, only one parameter has to be adapted. It is clear that in practice, a trade-off between the simplicity of the controller and the conservatism of the upperbound (that influences the controller magnitude) has to be made.

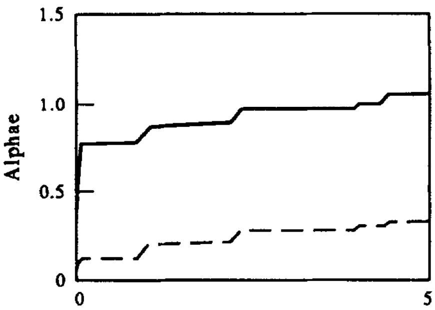

Fig. 3. Upperbounds estimates.

(ii) To illustrate the method, we present some simulation results for the case of the pendulum (see also Brogliato and Trofino Neto (1992) for more details on the dynamic model and the control input design). The simulations have been obtained on a MatLab package. q is the link angle. The numerical values are: $\varepsilon = 0.1$, $R_1 = \varepsilon$, $R_2 = 1.12\varepsilon$, the torque input disturbance $d(t) = 2\sin(15t)$, the mass $m = 10$, the length $l = 0.5$, the inertia $I = 0.2$, the desired angle trajectory is $q_d(t) = \cos(2t)$, $\tilde{q} = q - q_d$, the control parameters are chosen as $\lambda = 5$, $k = 10$, $a_e = 0.5$, $b_e = 50$, where the torque input is given by:

$$\tau = a_e(\ddot{q}_d - \lambda\dot{\tilde{q}}) + b_e\cos(q) - k(\dot{\tilde{q}} + \lambda\tilde{q}) + u. \quad (19)$$

Here we have chosen (note that this is neither the unique choice nor the simplest one, see the remark above):

$$\rho^T(v, t) = [([\ddot{q}_d - \lambda\dot{\tilde{q}}]^2 + \cos^2(q))^{1/2}, 1], \quad \alpha^{*T} = [\alpha_1^*, \alpha_2^*].$$

The results for $\tilde{q}$, $\hat{\alpha}$ and u are depicted in Figs 2–4. The different periods when $\hat{\alpha}$ is frozen clearly appear in Fig. 3. In Figs 5 and 6, $\tilde{q}$ and u are depicted in the case of a fixed upperbound (α_1^* and α_2^* are chosen equal to 1.5). The fixed parameters scheme provides a better value of the tracking error during the transient (the curve in Fig. 2 is smoother than the one in Fig. 5). We note also that the transient value of u in Fig. 4 is two times less than the one in Fig. 6. This can be explained by the fact that as the estimates are initialized at zero, u takes smaller values in the adaptive case until the estimates reach a larger value. After the transient peak, both control inputs u are close in magnitude, although it is smoother in the fixed parameter case.

(iii) The method presented in Liao *et al.* (1990) relies on the assumption that the nominal system (nonlinear invariant) is linearizable by static state feedback. In the case of rigid manipulators, this method will generally imply that the system parameters are known, except for special cases of manipulators (see Liao *et al.* (1990) for such an example). When the uncertainties vanish, the proposed scheme does not insure asymptotic stability. These, we believe, constitute the major differences between our scheme and the one in that paper.

(iv) The backstepping method used in Freeman and

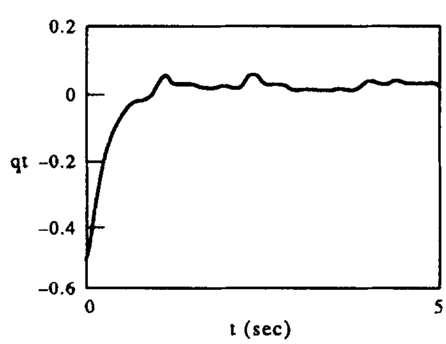

Fig. 2. Position tracking error (adaptive scheme).

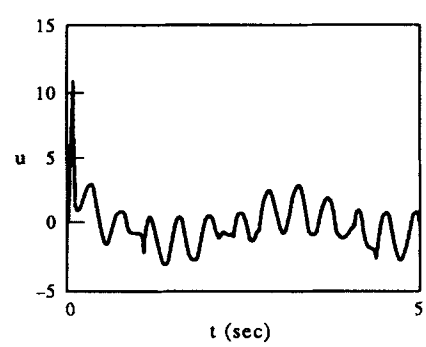

Fig. 4. Control input (adaptive scheme).

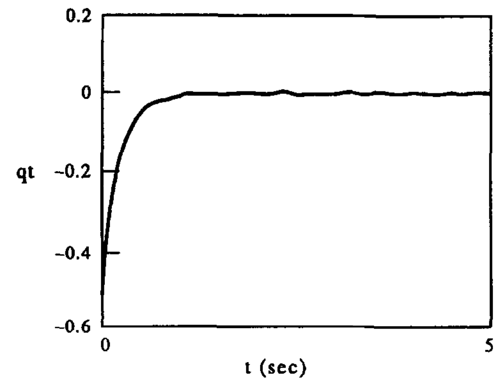

Fig. 5. Position tracking error (fixed parameters scheme).

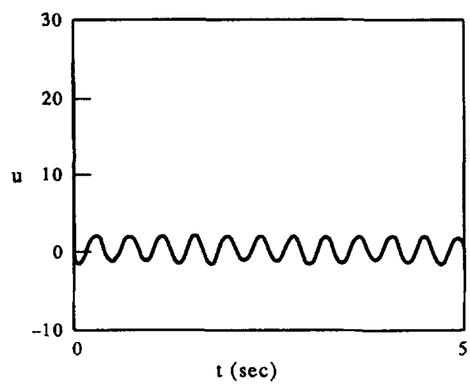

Fig. 6. Control input (fixed parameters scheme).

Kokotovic (1992) requires that the input be differentiable at each step. Thus, the algorithm presented in this paper cannot directly be extended to their method when more than two steps are needed, due to the discontinuities in the update law in (12).

(v) R_2 in (13) depends on γ_1 and γ_2, which in turn may depend on some system's parameters (see, for example, the case of rigid manipulators in Slotine and Li (1988)). This can be relaxed by replacing R_2 with some time-varying signal. (We refer the reader to Brogliato and Trofino Neto (1992) for a description of this modification.)

4. *Conclusions*

In this paper we have proposed a control algorithm for a class of systems which contain uncertainties. We have assumed that the so-called matching conditions are verified, but that the upperbound on the uncertainties is only partially known, i.e. it linearly depends on unknown constant parameters. It has been shown that the introduction of a dead zone with a suitable hysteresis in the update law of the estimates enables us to conclude the ultimate boundedness of the state of the system as well as boundedness of the estimates.

Contrary to some other attempts in the literature to relax the a priori knowledge on the uncertainty upperbound, the control scheme in this paper uses a continuous control law (with arbitrary boundary layer in the saturation function) and guarantees the convergence of the system state in a set around the origin whose size does not depend on the unknown upperbound. From this point of view, it can be seen as a direct extension of the work of Corless and Leitmann (1981). Further work should provide comparisons between this approach and other control methodologies, such as adaptive control. This, we believe, is not an easy task and will mainly depend on which type of nominal system we deal with, and the physical constraints (actuators, bandwidth, noise, unmodelled dynamics...) associated with it. A tentative comparison between a fixed parameters robust scheme and the adaptive scheme in Slotine and Li (1988) can be found in Spong (1992).

References

Barmish, B. R. and G. Leitmann (1982). On ultimate boundedness control of uncertain systems in the absence of matching assumptions. *IEEE Trans. Autom. Control,* **AC-27,** 153–158.

Barmish, B. R., I. R. Petersen and A. Feuer (1983). Linear ultimate boundedness control of uncertain dynamical systems. *Automatica,* **19,** 523–533.

Brogliato, B. and A. Trofino Neto (1992). Adaptive robust control of a class of nonlinear dynamic systems containing partially known uncertainties. In *Proc. Am. Control Conf.,* Chicago, IL, pp. 2559–2563.

Brogliato, B., A. Trofino and R. Lozano (1992). Robust adaptive control of a class of nonlinear first order systems. *Automatica,* **28,** 795–801.

Chen, Y. H. (1992). Adaptive robust control of uncertain systems with measurement noise. *Automatica,* **28,** 715–728.

Chen, Y. H. and G. Leitmann (1987). Robustness of uncertain systems in the absence of matching assumptions. *Int. J. Control,* **45,** 1527–1542.

Corless, M. (1993). Control of uncertain nonlinear systems. *ASME J. Dyn. Syst. Meas. Control,* **115,** 362–372.

Corless, M. and G. Leitmann (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamical systems. *IEEE Trans. Autom. Control,* **AC-26,** 1139–1144.

Corless, M. and G. Leitmann (1983). Adaptive control of systems containing uncertain functions and unknown functions with uncertain bounds. *J. Optimiz. Theory Applic.* **41,** 155–168.

Dawson, D. M., R. Genet and F. L. Lewis (1990). A hybrid adaptive learning controller for a robot manipulator. In *Proc. Symp. Adaptive/Learning Control,* **21,** pp. 51–54.

Dawson, D. M., Z. Qu and F. L. Lewis (1992). Hybrid adaptive robust control for a robot manipulator. *Int. J. ACASP,* **6,** 537–545.

Filipov, A. C. (1988). *Differential Equations with Discontinuous Right Hand Sides.* Kluwer, Dordrecht.

Freeman, R. A. and P. V. Kokotovic (1992). Backstepping design of robust controllers for a class of nonlinear systems. *NOLCOS,* Bordeaux, France, pp. 307–312.

Fu, L. C. (1992). Robust adaptive decentralized control of robot manipulators. *IEEE Trans. Autom. Control,* **AC-37,** 106–110.

Grayson, L. P. (1963). Design via Lyapunov's second method. In *Proc. 4th Joint Conf. on Autom. Control,* pp. 589–595.

Hsu, L. and R. Costa (1987). Bursting phenomena in continuous time adaptive systems with σ modification. *IEEE Trans. Autom. Control,* **AC-32,** 84–86.

Ioannou, P. A. and P. V. Kokotovic (1984). Instability analysis and improvement of robustness of adaptive control. *Automatica,* **20,** 583–594.

Johnson, G. W. (1964). Synthesis of control systems with stability constraints via the direct method of Lyapunov. *IEEE Trans. Autom. Control,* **AC-9,** 270–273.

Khalil, H. K. (1992). *Nonlinear Systems.* MacMillan, New York.

Leitmann, G. (1993). On one approach to the control of uncertain systems. *ASME J. Dyn. Syst. Meas. Control* **115,** 372–382.

Leung, T. P., Q. J. Zhou and C. Y. Su (1991). An adaptive variable structure model following control design for robot manipulators. *IEEE Trans. Autom. Control,* **AC-36,** 347–352.

Liao, T. L., L. C. Fu and C. F. Hsu (1990). Adaptive robust tracking of nonlinear systems and with an application to robotic manipulators. *Syst. Control Lett.* **15,** 339–348.

Marino, R. and P. Tomei (1992). Self-tuning stabilization of feedback linearizable systems. In *Proc. Int. Symp. on Adapt. Syst. Sign. Process.,* Grenoble, pp. 9–14.

Massera, J. L. (1956). Contributions to stability theory. *Ann. Math.,* **64,** 182–206.

Monopoli, R. V. (1965). Synthesis techniques employing the

direct method. *IEEE Trans. Autom. Control,* **AC-10,** 369–370.
Monopoli, R. V. (1966). Discussion on "two theorems on the second method. *IEEE Trans. Autom. Control,* **AC-11,** 140–141.
Peterson, B. B. and K. S. Narendra (1982). Bounded error adaptive control. *IEEE Trans. Autom. Control,* **AC-27,** 1161–1168.
Qu, Z., J. F. Dorsey, X. Zhang and D. M. Dawson (1991). Robust control of robots by the computed torque law. *Syst. Control Lett.* **16,** 25–32.
Ryan, E. P. and M. Corless (1984). Ultimate boundedness and asymptotic stability of a class of uncertain dynamical systems via continuous and discontinuous feedback control. *IMA J. Math. Control Inform.,* **1,** 223–242.
Singh, S. N. (1985). Adaptive model following control of nonlinear robotic systems. *IEEE Trans. Autom. Control,* **AC-30,** 1099–1100.
Slotine, J. J. and W. Li (1988). Adaptive manipulator control: a case study. *IEEE Trans. Autom. Control,* **AC-33,** 995–1003.
Sontag, E. (1990). *Mathematical Control Theory: Determinstic Finite Dimensional Systems.* Springer, Berlin.
Spong, M. W. (1992). On the robust control of robot manipulators. *IEEE Trans. Autom. Control,* **AC-37** 1782–1786.
Spong, M. W. and M. Vidyasagar (1989). *Robot Dynamics and Control.* Wiley, New York.
Yoo, D. S. and M. S. Chung (1992). A variable structure control with simple adaptation laws for upperbounds on the norm of the uncertainties. *IEEE Trans. Autom. Control,* **AC-37.**
Zinober, A. S. I. (ed.) (1990). Deterministic control of uncertain systems. *IEEE Control Engng Series,* **40.**

Appendix A

Definition of the ultimate boundedness (Barmish *et al.,* 1983). A solution $x(\cdot):[t_0, +\infty) \to R^n$, $\mathbf{x}(t_0) = \mathbf{x}_0$ of (1) is said to be uniformly and ultimately bounded with respect to a compact set X in R^n if there is a non-negative constant (time) $T(\mathbf{x}_0, X) < +\infty$, possibly dependent on $\mathbf{x}_0$ but not on t_0, such that $\mathbf{x}(t) \in X$ for all $t \geq t_0 + T(\mathbf{x}_0, X)$.

If this property holds for any initial condition, then the system is Globally Uniformly Ultimately Bounded (GUUB) with respect to X.

Appendix B

Examination of the closed-loop differential equations (1), (6) and (12) reveals that the main source of discontinuities in the closed-loop system is σ. Indeed, σ is defined as a function of the state and thus renders the right-hand side of the overall closed-loop system discontinuous with respect to the system state vector, although the discontinuities do not enter directly in the control input (6) which is a continuous-time function. This is the reason why we have to take care of the existence of solutions. In the first part of the proof, we show that the discontinuities in σ, if any, cannot occur infinitely often in a finite-time interval. Thus, a unique and uniformly bounded solution exists for the closed-loop system on each of the time intervals between two consecutive discontinuities. Those solutions can be concatenated to obtain a unique and uniformly bounded solution until a finite time $T > t_0$ is reached such that $\hat{\alpha}(T) \geq \alpha^*$.† The second part of the proof is devoted to showing that as soon as $\hat{\alpha}$ is greater than α^*, then there exists a finite time $T' > T$, such that for all $t \geq T'$, $\mathbf{x}(t)$ cannot escape from the set $V_{\mathbf{x}}^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ and hence from $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ (see (14)), i.e. $\mathbf{x}$ is uniformly and ultimately bounded and $\hat{\alpha}$ is frozen at a bounded value. This enables us to conclude that the solution can be extended on $[t_0, +\infty)$.

B.1. Existence, uniqueness and boundedness of solutions. First note from (12) and Fig. 1 that $\hat{\alpha}$ is not decreasing. We assume, without loss of generality, that $|\mathbf{x}(t_0)| = R > R_1$, i.e. the system is initialized outside the ball B_{R_1}, and $\sigma(t_0) = 1$.

† For $x \in R^p$, $y \in R^p$ we note $x < y$ if $x_i < y_i$ for at least one $i \in \{1, p\}$, and $x \geq y$ if for all $i \in \{1, p\}$ we have $x_i \geq y_i$.

Definition. Let us define $D = [t_0, t_1) \times B$ as a bounded domain for $\{t, \bar{\mathbf{x}}\}$ where $t \in [t_0, t_1)$ and $\bar{\mathbf{x}} \in B$ for some closed connected set $B \subset R^{n+p}$. t_1 is defined in such a way that no discontinuity has occurred in $\hat{\alpha}$. ∎

We assume that $\hat{\alpha}(t_1) < \alpha^*$, i.e. the estimate has not reached the upper bound at t_1. In view of the definition of the bounded domain D and assumption A2, it follows that for any initial condition in D, a unique solution exists‡ in a certain neighbourhood of $(t_0, \bar{\mathbf{x}}(t_0))$, say $D_1 = [t_0, t_1') \times B_r$, for some $t_1' > t_0$ and $r \geq R$. Now, it follows from (15) and (17) that as long as $\mathbf{x}(t)$ lies outside B_{R_1} one gets

$$\dot{V}(\mathbf{x}(t), \tilde{\alpha}(t), t) \leq -\varepsilon < 0, \quad \text{(B.1)}$$

so that

$$0 \leq \gamma_1(|\mathbf{x}|) \leq V(\mathbf{x}, \tilde{\alpha}, t) < V(\mathbf{x}(t_0), \alpha^*, t_0) \leq \gamma_2(|\mathbf{x}(t_0)|) + \alpha^{*\mathrm{T}}\alpha^*, \quad \text{(B.2)}$$

($\tilde{\alpha}^{\mathrm{T}}(t_0)\tilde{\alpha}(t_0) \leq \alpha^{*\mathrm{T}}\alpha^*$ as $\hat{\alpha}(t_0) \geq 0$).

Therefore, it follows that during this interval of time:

$$|\mathbf{x}(t)| \leq \gamma_1^{-1}(\gamma_2(|\mathbf{x}(t_0)|) + \alpha^{*\mathrm{T}}\alpha^*). \quad \text{(B.3)}$$

Thus, one sees that $\mathbf{x}$ (and consequently $\hat{\alpha}$) are uniformly bounded on D_1, i.e. the right-hand side of (B.3) does not depend either on t_0 nor on t_1. Moreover, the bound is independent of the size of B_r, i.e. B_r can be made arbitrarily large while (B.3) still holds. This enables us to extend the solution up to the boundary of D, i.e. until the first discontinuity in $\hat{\alpha}$ occurs. (Note, from (7) and (B.1) that there exists a finite time t_1 such that $|\mathbf{x}(t_1)| = R_1$.) This solution, which is uniformly bounded on D, is unique (see Filipov (1988)). Now, there exists a time t_2 such that for all $t \in [t_0, t_2)$: $|\mathbf{x}(t)| \leq R_2$. On $[t_0, t_2)$, $\hat{\alpha}$ is frozen and we have $\hat{\alpha}(t) = \hat{\alpha}(t_1)$. Since $\hat{\alpha}(t_1) < \alpha^*$, it may happen that $\mathbf{x}(t)$ escapes from the set $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ at the time t_2. Indeed, the classical reasoning employed in Corless and Leitmann (1981) cannot be repeated here. From (14), we conclude that one cannot insure that $\mathbf{x}(t)$ will remain inside $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$.

B.1.1. Extension of the solution. Note, that the above reasoning can be repeated to show the existence and uniqueness of a solution of the closed-loop system in the interval (t_1, t_2). Then, we can define a solution on $[t_0, t_2)$ by concatenation of the solutions on $[t_0, t_1)$ and (t_1, t_2). The first solution starts at the initial condition and ends when the first discontinuity in $\hat{\alpha}$ occurs. The second solution starts at the final value of the first solution and ends when the second discontinuity at t_2 occurs. It has also to be emphasized that the discontinuities cannot occur infinitely often in a finite-time interval (i.e. the measure of the interval $[t_1, t_2]$ is strictly positive). Indeed, with $\mathbf{x}(t)$ being uniformly bounded on this interval, it follows from assumption A2 that $\dot{\mathbf{x}}$ is bounded also, so that infinitely fast dynamics are avoided. (Note that γ_1 and γ_2 in (3) can always be chosen in a way such that $R_2 - R_1 > 0$.)

Now for some $t > t_2$, $\mathbf{x}(t)$ lies outside the set $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$. During this period we have

$$V(\mathbf{x}, \tilde{\alpha}, t) < V(\mathbf{x}(t_2), \tilde{\alpha}(t_2), t_2) \leq \gamma_2(\gamma_1^{-1} o \gamma_2 o \gamma_3^{-1}(2\varepsilon + h)) + \tilde{\alpha}^{\mathrm{T}}(t_1)\tilde{\alpha}(t_1), \quad \text{(B.4)}$$

because (B1) holds while $\hat{\alpha}$ increases. Therefore we get

$$|\mathbf{x}(t)| \leq \gamma_1^{-1}(\gamma_2 o \gamma_1^{-1} o \gamma_2 o \gamma_3^{-1}(2\varepsilon + h)) + \tilde{\alpha}^{\mathrm{T}}(t_1)\tilde{\alpha}(t_1), \quad \text{(B.5)}$$

for all $t \in (t_2, t_3)$, where t_3 is defined as the finite time such that $\mathbf{x}(t_3) = R_1$, i.e. $\mathbf{x}(t)$ re-enters the ball B_{R_1}. The reasoning, above, applies as long as $\hat{\alpha}$ remains smaller than α^*. Repeating the arguments in (B.3) and (B.5), one sees that on each time interval (t_i, t_{i+1}) between two consecutive discontinuities (where t_i and t_{i+1} are such that σ switches from 0 to 1 at t_i and from 1 to 0 at t_{i+1}) we get

$$|\mathbf{x}(t)| \leq \gamma_1^{-1}(\max[\gamma_2 o \gamma_1^{-1} o \gamma_2 o \gamma_3^{-1}(2\varepsilon + h), \gamma_2(|\mathbf{x}(t_0)|] + \tilde{\alpha}^{\mathrm{T}}(t_i)\tilde{\alpha}(t_i)). \quad \text{(B.6)}$$

‡ Indeed the closed-loop system appears as a Caratheodory equation on D; see also Filipov (1988).

Note from (12), that the successive values $\hat{\alpha}(t_i)$ are bounded, because $\hat{\alpha}$ is increased by a finite amount on (t_i, t_{i+1}).

B.2. Global uniform ultimate boundedness of $\mathbf{x}(t)$. Ultimate boundedness of $\mathbf{x}$ hinges mainly on the fact that the total time during which $\hat{\alpha}$ is updated (i.e. the total time during which the system is in cases C1 or C3) is finite. First notice from (15) and (17) that cases C1 and C3 are 'unstable' in the sense that each time the system is in one of these cases, then V strictly decreases and $\mathbf{x}$ reaches the ball B_{R_1} in finite time, so that the system is in case C2 or C4. Now from (16) and (18), we see that nothing guarantees that the system will remain in C2 or C4 unless $\hat{\alpha} \geq \alpha^*$. Let us first analyse what happens when the system is in C1: then $\hat{\alpha}$ is strictly increasing, and during this time we have (see assumption A1)

$$\dot{\hat{\alpha}}_i \geq \beta > 0 \quad \text{for some } \beta \text{ and } i \in \{1, p\}. \tag{B.7}$$

Let Ω be defined as $\Omega = \{t \geq t_0 / \dot{\hat{\alpha}}(t) > 0\}$, i.e. Ω is the total time during which $\hat{\alpha}$ increases (i.e. at least one component of $\hat{\alpha}$ increases). Assume that Ω has a large enough measure so that for some $T > t_0 : \hat{\alpha}(T) \geq \alpha^*$. Let us denote Ω_T the truncation of Ω up to time T. Note from (B.7) that Ω_T has a finite measure†

$$\mu[\Omega_T] \leq \max_{i \in \{1,p\}} \frac{\alpha_i^*}{\beta}. \tag{B.8}$$

(We recall that $\hat{\alpha}(t_0) \geq 0$.) We shall see that this implies that Ω has also a finite measure, i.e. the total time during which the trajectory of the closed-loop system lies outside the set $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ is finite. Indeed, assume that T has been reached. Then for all $t \geq T$, $\hat{\alpha}(t) \geq \alpha^*$. Moreover $\hat{\alpha}$ increases until the ball B_{R_1} is reached at some time T'. Then, $\hat{\alpha}$ is frozen. Assume that $\mathbf{x}(t)$ escapes from B_{R_1}, then as $\tilde{\alpha} > 0$ we get

$$\dot{V}(\mathbf{x}, \tilde{\alpha}, t) = \dot{V}_{\mathbf{x}}(\mathbf{x}, t) \leq -\gamma_3(|\mathbf{x}|) - |g(\mathbf{x}, t)|\, \rho^T(\mathbf{x}, t)\tilde{\alpha}$$
$$\leq -\gamma_3(|\mathbf{x}|). \tag{B.9}$$

This means that we can repeat the classical reasoning when the upperbound is fixed (see, e.g. Corless and Leitmann (1981)), choosing for example $t = T'$ as an initial time (i.e. we consider the convergence analysis with known and fixed upperbound $\hat{\alpha} \geq \alpha^*$, $\mathbf{x}$ being initialized in the ball B_{R_1}). It follows that $\mathbf{x}(t)$ can no longer escape from the set $V_{\mathbf{x}}^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ and hence from $\gamma_1^{-1}[\gamma_2 o \gamma_3^{-1}(2\varepsilon + h)]$ (see (14)). Therefore, $\hat{\alpha}$ is frozen and we have $\hat{\alpha}(t) = \hat{\alpha}_f$ for all $t \geq T'$, and Ω has a finite measure.

The case when the system is in C3 is slightly different as it is not guaranteed then that $\hat{\alpha}$ strictly increases. Indeed $g(\mathbf{x}, t)$ can be zero, and $\hat{\alpha}$ may be stuck at a value smaller than α^*. Let us examine this case in more detail. Two situations can happen: either there exists a finite time T such that for all $t \geq T$, $g(\mathbf{x}, t)$ is zero almost everywhere, in which case $\hat{\alpha}$ is frozen for $t \geq T$ and from (9) we conclude that $\mathbf{x}$ asymptotically converges towards zero. (In this situation, the perturbation enters in the system tangentially to the level sets of $V_{\mathbf{x}}$ almost everywhere and thus does not influence the variation of $V_{\mathbf{x}}$.); or there exist sequences $[t_i, t_{i+1}]$, $i \in N$, $t_i \to +\infty$ as $i \to +\infty$, of strictly positive measure such that $g(\mathbf{x}, t) \neq 0$ on $[t_i, t_{i+1}]$. Then, we see that the total time spent by the system in C3 is finite using the same arguments as above.

Finally, we can state that $\Omega = \{t \geq t_0 / \dot{\hat{\alpha}}(t) > 0\}$ has a finite measure, i.e. the total time during which $\mathbf{x}$ lies outside the ball B_{R_2} is finite. Then, using the same argument as above we conclude the proof. Moreover we have for all $t \geq t_0$

$$|\mathbf{x}(t)| \leq \gamma_1^{-1}\Big(\max\,[\gamma_2 o \gamma_1^{-1} o \gamma_2 o \gamma_4^{-1}(\varepsilon),\ \gamma_2(|\mathbf{x}(t_0)|)] + \max_{t_i < T'} \tilde{\alpha}^T(t_i)\tilde{\alpha}(t_i)\Big). \tag{B.10}$$

Now consider the domain $D_2 = [t_0, T'] \times B$ and let the ball B become arbitrarily large; since we have shown that $\bar{\mathbf{x}}$ is uniformly bounded on t_0, T' and cannot escape from a compact subset of $R^n \times R^p$ for all $t \geq T'$, we finally conclude that the solutions can be extended on $D = [t_0, +\infty) \times B$ and are uniformly bounded on D (see, e.g. Khalil (1992).

† This type of result seems to be generic when dead zones are used in parameter adaptive control (see e.g. Peterson and Narendra (1982), Brogliato *et al.* (1992), although the context and the arguments used in these references to prove finite measure of the total adaptation time are different from here).