On the Control of Complementary-Slackness Juggling Mechanical Systems

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***Abstract*—This paper studies the feedback control of a class of complementary-slackness hybrid mechanical systems. Roughly, the systems we study are composed of an uncontrollable part (the "object") and a controlled one (the "robot"), linked by a unilateral constraint and an impact rule. A systematic and general control design method for this class of systems is proposed. The approach is a nontrivial extension of the one degree-of-freedom (DOF) juggler control design. In addition to the robot control, it is also useful to study some intermediate controllability properties of the object's impact Poincaré mapping, which generally takes the form of a nonlinear discrete-time system. The force input mainly consists of a family of dead-beat feedback control laws, introduced via a recursive procedure, and exploiting the underlying discrete-time structure of the system. The main goal of this paper is to highlight the role of various physical and control properties characteristic of the system on its stabilizability properties and to propose solutions in certain cases.**

***Index Terms*—Complementary slackness, feedback, hybrid, impact Poincaré mappings, nonsmooth, underactuated, viabililty.**

I. Introduction

A. General Introduction

RECENT researches in the robotics and the systems and control communities on mechanical systems subject to unilateral constraints have focused on stabilization of manipulators during complete robotic tasks [10], [39], well-posedness and system theoretic issues [9], [30]–[32], walking and hopping machines [1], [5], [7], [8], [12], control of juggling and catching robots [6], [26], [27], [33], [37], systems with dynamic backlash [2], [16], stabilization of polyhedral objects in some manipulation tasks [35], and nonprehensile manipulation [11], [19], [24]; see [19] for a more complete bibliography on this last topic. The work presented in this paper focuses essentially on the last five listed topics. The open-loop models used are basically rigid body dynamics with a set of unilateral constraints on the generalized position. Such hybrid dynamical systems may be represented as follows:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = Tu + \nabla_q h(q, t)\lambda \tag{1}$$

$$h(q, t) \geq 0 \tag{2}$$

$$\lambda \geq 0, \qquad \lambda^T h(q, t) = 0 \tag{3}$$

where the classical dynamics of Lagrangian systems is in (1), the set of unilateral constraints is in (2) with $h(q, t) \in \mathbb{R}^m$, and the so-called complementarity conditions [14] are in (3), where $\lambda \in \mathbb{R}^m$ is the Lagrange multiplier vector. We assume frictionless constraints. $q \in \mathbb{R}^n$, $M(q) \in \mathbb{R}^{n\times n}$, $C(q, \dot{q})\dot{q} \in \mathbb{R}^n$, $g(q) \in \mathbb{R}^n$, and $u \in \mathbb{R}^{n_u}$ are the generalized coordinate vector, the inertia matrix, the Coriolis and centrifugal terms, the gravity forces and torques, and the control input, respectively, and $T \in \mathbb{R}^{n\times n_u}$. In order to render the dynamical system complete, we must add to (1)–(3) a so-called restitution law that relates postimpact and preimpact velocities. Such physical rules are necessary [30], [40]. The most widely used restitution rule is known as Newton's conjecture [4]. It is based on the knowledge of restitution coefficients e_i. When $h(q, t) = h(q)$, it is represented in its generalized form as follows:

$$\dot{q}(t_k^+)^T \nabla_q h_i(q(t_k)) = -e_i \dot{q}(t_k^-)^T \nabla_q h_i(q(t_k)) \tag{4}$$

where $\nabla_q h_i \in \mathbb{R}^n$, $1 \leq i \leq m$, and t_k generically denote the impact times (the superindices $+$ and $-$ stand, respectively, for the instants just after and just before the collisions), with $\dot{q}(t_k^-)^T \nabla_q h_i(q(t_k), t_k) < 0$. In case of a codimension 1 constraint ($m = 1$), $e \in [0, 1]$ from energetical arguments. The system in (1)–(4) is complete, in the sense that, given preimpact velocities $\dot{q}(t_k^-)$, we are able to calculate the postimpact velocities and continue the integration after the collision has occurred. Notice that the system in (1)–(4) is a complex hybrid dynamical system [32]. This class of dynamical systems can be divided further into subclasses. In particular, the case in which the free-motion dynamics are controllable has recently received attention [31], [39]. This class does not, however, cover some impacting robotic systems. Indeed, write the dynamical equations of a two degree-of-freedom (DOF) juggler; i.e., a system composed of an object (a point mass) subject to gravity, which rebounds on a controlled table, as shown in Fig. 1:

$$m_1\ddot{x} = -\lambda_1 \sin(\theta+\alpha) - \lambda_2 \sin(\theta-\alpha) \tag{5}$$

$$m_1\ddot{y} = -m_1 g + \lambda_1 \cos(\theta+\alpha) + \lambda_2 \cos(\theta-\alpha) \tag{6}$$

$$\begin{aligned} I_2\ddot{\theta} = {} & u_1 - \lambda_1[(y-Y)\sin(\theta+\alpha) + x\cos(\theta+\alpha)] \\ & - \lambda_2[(y-Y)\sin(\theta-\alpha) + x\cos(\theta-\alpha)] \end{aligned} \tag{7}$$

$$m_2\ddot{Y} = u_2 - \lambda_1 \cos(\theta+\alpha) - \lambda_2 \cos(\theta-\alpha) \tag{8}$$

$$\begin{cases} h_1(x, y, Y, \theta) = (y-Y)\cos(\theta+\alpha) - x\sin(\theta+\alpha) \geq 0 \\ h_2(x, y, Y, \theta) = (y-Y)\cos(\theta-\alpha) - x\sin(\theta-\alpha) \geq 0 \end{cases}$$

$$\lambda^T h(x, y, Y, \theta) = 0, \quad \lambda \geq 0 \tag{9}$$

$$\text{Restitution rule in (4).} \tag{10}$$

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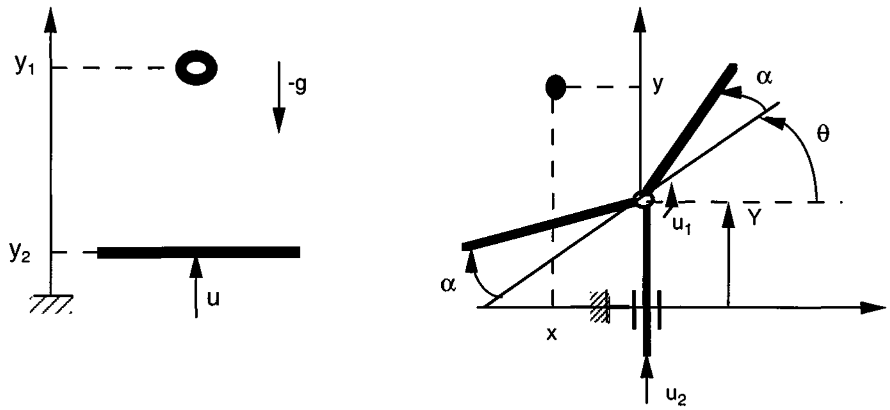

Fig. 1. One and two DOF jugglers.

Notice that the ball dynamics in (5) and (6) are not controlled because only gravity acts on the ball. Hence, the only way to influence the trajectory of the ball is through impacts. This is a strong motivation for considering the feedback control of a two DOF juggler, as in (5)–(10), because it represents a simplified model of manipulation of objects through "controlled" collisions with a robotic device [35]. We may suppose that the length of the surface is infinite. Hence, the impacts always occur with the same side of the table. In this paper, we shall use this example as an illustration of the influence of the physical and control parameters on the control design.

B. A Class of Complementary-Slackness Hybrid Systems

Juggling robots may also be considered as a particular case of a class of complementary-slackness systems

$$\dot{z}_1 = f_1(z_1, t, \lambda) \tag{11}$$

$$\dot{z}_2 = f_2(z_2, u, \lambda) \tag{12}$$

$$h(q_1, q_2, t) \geq 0, \quad \lambda^T h(q_1, q_2, t) = 0, \quad \lambda \geq 0 \tag{13}$$

where $u \in \mathbb{R}^{n_u}$, $z_i \triangleq (q_i^T, \dot{q}_i^T)^T \in \mathbb{R}^{n_i}$, $i = 1, 2$ (n_1 and n_2 may have the same or different values), and a restitution rule has to be added to complete the model. Notice that if $\lambda \in \mathbb{R}^m$, such a system may *a priori* evolve in 2^m different modes [32]: its hybridness is therefore intrinsic. As indicated in the title and the abstract, we choose to generically name such systems *jugglers*, even though, as we shall see later, other types of systems may fall into this class. In this setting, (11) plays the role of the "object," and (12) is the "robot." $h(\cdot, \cdot, \cdot)$ may represent the "distance" between the object and the robot, but this is not always the case, as some examples will prove. A necessary condition to transform (1) into (11) and (12) is that a generalized coordinate transformation $q = Q(q_1, q_2)$ exists and fulfills $(\partial Q/\partial q_1^i)^T M(q)(\partial Q/\partial q_2^j) = 0$ for all $i \in \{1, \cdots, n_1\}$, $j \in \{1, \cdots, n_2\}$. Also consider $\lambda \equiv 0$, and notice that the system in (11) and (12) corresponds to (1) written in a noncontrollable canonical form. Such systems are therefore quite different from so-called triangular systems [15]. In particular, setting $\lambda \equiv 0$, (11) and (12) are not in general locally feedback stabilizable because the uncontrollable modes z_1 are not necessarily stable. The control problem for (11)–(13) may change depending on our goal: for instance, in a system with clearance, like the impact damper [2], [16], we may consider that (11) creates disturbances in the dynamics of (12), or on the contrary, we may desire to control (11) using the impacts. In this paper, we will mainly focus on the control of z_1 from the input u in (12), via sequences of impacts. In other words, we shall restrict ourselves to control tasks with zero-measure contact phases. The controllability and stabilizability properties of such hybrid systems, which depend on the vector fields in (11) and (12) and on h in (13), have not yet been fully understood. It is advocated here that impact Poincaré mappings [40] provide a suitable framework for such analysis. Let us introduce the following definition.

Definition 1: A *viable* controller u is a function $u(q, \dot{q}, t)$ such that 1) $h(q, t) > 0$ or $\{h(q(\tau_0), \tau_0) = 0$ and $\dot{q}(\tau_0^-)^T \nabla_q h(\tau_0) = 0\}$ between programmed impact times, and 2) $z_1(t)$ and $z_2(t)$ are right-continuous of local bounded variation (LBV) in time.

Here, programmed impacts are defined as those shock instants planned in the control design (contrary to accidental impacts that may be caused by various model uncertainties or disturbances in the control loop). As we shall see, the sequence $\{t_k\}_{k \in \mathbb{N}}$ depends on the controller design that in turn must incorporate the object ballistic constraints. Let us recall that an LBV function possesses a countable set of discontinuity points [21]; hence, a viable input ensures the well-posedness of the closed-loop system.

Definition 2: The "object" in (11) is controllable if given $t, T, T > t, T$ satisfying the object's ballistic constraints, $z_1(t)$ and $z_2(t)$, such that $h(q_1(t), q_2(t), t) \geq 0$, and $z_1(T)$, a viable control law u that drives z_1 from $z_1(t)$ to $z_1(T)$ exists, with $h(q_1(T), q_2(T), T) \geq 0$.

The study of such notions, however, requires some intermediate steps, for which some basic properties are important (interestingly enough, some of them are similar to those done in [17] for the control of a class of cascaded nonlinear systems that model some nonholonomic mechanical systems). Among those:

1) the properties of the object's flow $\phi_{z_1}(t;\ t_k, z_1(t_k^+)) = z_1(t)$ on (t_k, t_{k+1}) [explicit knowledge of the trajectories, or controllability with state $z_1(t)$ and input $z_1(t_k^+)$];
2) the controllability of (12);
3) the controllability of (11) through impacts;

4) the relative degree[1] r^* of h with respect to u and the relative degrees r^λ of h with respect to λ;
5) n_1, n_2 and n_u;
6) the boundedness of z_1 between impacts or the ability of the robot to extracting energy to the object;
7) the restitution rule (i.e., the value of the restitution coefficient e, the form of the constraint $h(\cdot, \cdot, \cdot)$, the codimension m_c of the striked subspace in case of multiple shocks).

To illustrate the so-called ballistic constraints in Definition 2, consider for instance the one DOF juggler in the next section, with $y_1(T) < y_1(t)$: clearly, we must wait for the ball to fall before hitting it, and the required time depends only on the vector field f_1. The interest for considering systems as in (11)–(13) is that their study finds potential applications in all types of juggling robots, catching tasks, nonprehensile manipulation (pushing-and-striking tasks with a workpiece free to slide on a work surface, which frequently occur in robotic applications [24]), stabilization of manipulators on passive dynamical environments (some "hammer-like" tasks are possible), and platoons of carts with a leading cart. It is also noteworthy that walking and hopping machines may be written in a similar form as in (11)–(13), where (11) may represent the dynamics of the mass center when all of the feet are detached from the ground; see, e.g., [1] and [7], as well as rocking-block-like models of buildings excited by earthquakes [40], whose active control is a topic of research [18]. Finally, models of systems with dynamic backlash also fit within the framework in (11)–(13) (e.g., the impact damper; see [40] and references therein). Therefore (11)–(13) constitute a large class of dynamical systems that deserves close attention and whose control is a challenging problem, as many of the above-cited references witness.

Contrary to most of the previous works on the topic, we shall consider the full dynamics of the system (11)–(13). We are therefore interested in designing directly the controller u that is to be implemented on the robot and to propose a general control design framework for complementary-slackness juggling mechanical systems. In Section II, we briefly recall the controller and the closed-loop analysis for the one DOF juggler that constitutes the basic benchmark example of complementary-slackness juggling systems. Controllability concepts based on the study of some impact Poincaré mappings associated with the object are introduced, which are thought to be useful for the overall control design. Section III presents the control strategy and the closed-loop analysis for a more general class of complementary-slackness juggling systems. The codimension one $(m = 1)$ case is analyzed first. Then, the multiconstraint case is examined. Conclusions are given in Section IV. The relationships with various published results are pointed out throughout the paper.

II. One DOF Juggler

In this section, we briefly recall the results presented in [37]. The proofs can be found in those references. As we shall demonstrate in the next section, this benchmark example does not encapsulate the whole essence of the general problem. It provides us, however, with a nice starting point.

Lemma 1: Consider the dynamics of a one DOF juggler

$$m_1\ddot{y}_1 + m_1 g = \lambda \tag{14}$$

$$m_2\ddot{y}_2 + m_2 g = u - \lambda \tag{15}$$

$$y_1 - y_2 \geq 0, \quad \lambda(y_1 - y_2) = 0, \quad \lambda \geq 0 \tag{16}$$

$$\dot{y}_1(t_k^+) - \dot{y}_2(t_k^+) = -e[\dot{y}_1(t_k^-) - \dot{y}_2(t_k^-)], \quad e \in [0, 1]. \tag{17}$$

Suppose that $x(0)$ and u are such that an impact or a contact time t_0 exists. Define the following control input:

$$u = m_2 g + m_2 v \tag{18}$$

$$v = A_k(t - t_k) + B_k \tag{19}$$

$k \geq 0$, with

$$A_k = \frac{6}{d_k^2}\left(\dot{y}_2^*(k+1) + \dot{y}_2(t_k^+)\right) - \frac{12}{d_k^3}\left(y^*(k+1) - y(k)\right) \tag{20}$$

$$B_k = -\frac{2}{d_k}\left(\dot{y}_2^*(k+1) + 2\dot{y}_2(t_k^+)\right) + \frac{6}{d_k^2}\left(y^*(k+1) - y(k)\right) \tag{21}$$

$$d_k = \frac{\dot{y}_1(t_k^+) + \sqrt{\dot{y}_1^2(t_k^+) - 2g\left(y^*(k+1) - y(k)\right)}}{g} \tag{22}$$

$$\begin{aligned}\dot{y}_2^*(k+1) = &\ \frac{1+m}{1+e} \\ = &+\frac{m-e}{1+e}\sqrt{\dot{y}_1^2(t_k^+) - 2g\left(y^*(k+1) - y(k)\right)}\end{aligned} \tag{23}$$

where $y(k) \triangleq y_1(t_k) = y_2(t_k)$, and $y^*(k+1)$ and $\dot{y}_1^*(k+1)$ are chosen such that

$$y^*(k+1) < h_k \triangleq \begin{cases} y(k) + \dfrac{\dot{y}_1^2(t_k^+)}{2g}, & \text{if } \dot{y}_1(k) > 0 \\ y(k), & \text{if } y_1(k) \leq 0 \end{cases} \tag{24}$$

and

$$\dot{y}_1(t_k + d_k) - \dot{y}_2^*(k+1) < 0. \tag{25}$$

Then, for all $k \geq 0$

1) $y_1(t) - y_2(t) > 0, \forall\, t \in (t_k, t_k + d_k)$;
2) $t_{k+1} = t_k + d_k$;
3) $y(k+1) = y^*(k+1)$;
4) $\dot{y}_1(t_{k+1}^+) = \dot{y}_1^*(k+1)$.

Remark 1: The control input force (18)–(21) is based on a dead-beat control strategy (inversion of the controllability Grammian). Other dead-beat inputs can be derived by simply adding position y_2 or velocity $\dot{y}_2$ feedback to the robot, but they may not be viable inputs. Such controllers are basically open loop. We notice from (18)–(25), however, that the controller is computed from the value of the state at t_k^+ (which does not necessarily mean that the states are measured *at* t_k^+). Property 1) may be used to perform an indirect measurement by measuring for instance the apex of the object's orbit. Consequently, u is a state feedback for the system considered as a discrete-time operator at the shock times t_k. •

From property 1), it is clear that we may replace the dynamics in (14) by any integrable vector field. Difficulties in calculating the flight times may occur, however, (hence, d_k), for instance, if

[1] As defined in [23].

some damping is added on the object. In practice, we may use a numerical estimation of d_k or study conditions under which the presented scheme is robust with respect to such uncertainty. As shown in [37], the next step is to define the signals $y^*(k+1)$ and $\dot{y}_1^*(k+1)$, which can be regarded as the desired trajectory of the ball, whereas $\dot{y}_2^*$ in (23) is the desired robot's preimpact trajectory. We still consider the system at a generic impact time t_k.

Lemma 2: Let $y^*(k+1)$ and $\dot{y}_1^*(k+1)$ be given as

$$y^*(k+1) = \begin{cases} y_d, & \text{if } h_k > y_d \\ y(k)+r, & \text{if } h_k \le y_d \end{cases} \tag{26}$$

$$\dot{y}_1^*(k+1) = \begin{cases} \dot{y}_d, & \text{if } h_k > y_d \\ \sqrt{\dot{y}_d^2 + 2g\,(y_d - y(k) - \gamma)}, & \text{if } h_k \le y_d \end{cases} \tag{27}$$

where $\dot{y}_d > 0$, and γ is such that

$$y(k) + \gamma < h_k. \tag{28}$$

Then, (x_k, Δ_k) converges toward its desired value $(x_d, \Delta_d) = (y_d, y_d, \dot{y}_d, ((1-e-2em)/(1+e))\dot{y}_d, (2/g)\dot{y}_d)$ after at most two impacts, i.e., $(x_{k+i}, \Delta_{k+i}) = (x_d, \Delta_d)\ \forall i \ge 2$.

Notice that the logic conditions in (26)–(28) are necessary because of the ballistic constraints imposed by the object's motion, and they justify the difference between the desired values $(\cdot)_d$ and those denoted as $(\cdot)^*$. Such control strategy has been shown to present several nice properties [37]: robustness with respect to various uncertainties (measurement noise, restitution coefficient), possibility of modification of $\dot{y}_1^*$ and y_1^* to cope with control saturations, and flexibility to cope with various object's orbits.

Remark 2 [Three-Steps Recursive Control Design]: Describe a systematic, recursive control design method, which will enable us to recover the dead-beat strategy in (18)–(27) as well as new controllers. Let us choose a structure for $v(t)$ as $v(t) = A_k\Delta + B_k$, $\Delta = t - t_k$, with A_k and B_k constant on (t_k, t_{k+1}). Integrating (14) between impacts or using (17), we get

$$\begin{cases} y_1(t_{k+1}) = y_1(t_k) + \dot{y}_1(t_k^+)\Delta_k - \dfrac{g}{2}\Delta_k^2 \\ \dot{y}_1(t_{k+1}^+) = \dfrac{m-e}{1+m}(\dot{y}_1(t_k^+) - g\Delta_k) + \dfrac{1+e}{1+m}\dot{y}_2(t_{k+1}^-) \end{cases} \tag{29}$$

and

$$\begin{cases} y_2(t_{k+1}) = y_2(t_k) + \dot{y}_2(t_k^+)\Delta_k + A_k\dfrac{\Delta_k^3}{6} + B_k\dfrac{\Delta_k^2}{2} \\ \dot{y}_2(t_{k+1}^-) = \dot{y}_2(t_k^+) + A_k\dfrac{\Delta_k^2}{2} + B_k\Delta_k \end{cases} \tag{30}$$

with $m = m_1/m_2$.

- *Step 1:* We choose Δ_k and $\dot{y}_2(t_{k+1}^-)$ as the inputs of the system in (29), such that $y_1(t_{k+1}) = y_1^*(k+1)$ and $\dot{y}_1(t_{k+1}^+) = \dot{y}_1^*(k+1)$. This step gives

$$\begin{aligned} (29)\ &\Leftrightarrow F_1(y_1^*(k+1), \dot{y}_1^*(k+1), \Delta_k, \dot{y}_2(t_{k+1}^-)) = 0 \\ &\Rightarrow \begin{cases} \Delta_k = d_k \\ \dot{y}_2(t_{k+1}^-) = \dot{y}_2^*(k+1) \end{cases} \end{aligned} \tag{31}$$

  where d_k and $\dot{y}_2^*(k+1)$ are given in Lemma 1, and for some function F_1.
- *Step 2:* Introducing the values in (31) into (30), we get $F_2(d_k, \dot{y}_2^*(k+1), A_k, B_k) = 0$ for some function F_2, from which we deduce A_k and B_k, which are equal to those in Lemma 1.
- *Step 3:* Check viability, i.e., the sign of the function $h(\Delta) \triangleq y_1(t) - y_2(t)$ on (t_k, t_{k+1}).

It is noteworthy that the success of Steps 1 and 2 relies on the invertibility properties of the first "subsystem" [which is not to be confused with the mapping in (36) that is obtained assuming the *a priori* knowledge of an input u satisfying properties 1)–3) in the introduction of Section III] with Δ_k and $\dot{y}_2(t_{k+1}^-)$ as inputs, i.e., on the existence of solution to the algebraic equations $F_1 = 0$ and $F_2 = 0$. We can choose another controller structure $v(t)$. •

Remark 3:

- The use of open-loop controllers during flight times $[t_k, t_{k+1})$ is further motivated by the results on control holdability of sets of the form $\{q\colon\ Cq \ge 0\}$ [30]. It is shown in [30, Corollaries 5.6.2 and 7.4.9] that, for linear mechanical systems $M\ddot{q} + C\dot{q} + Kq = Tu$, such sets are neither positively invariant (with $T = 0$) nor closed-loop holdable by static or dynamic feedback. (i.e., $\nexists F \in \mathbb{R}^{n_u \times 2n}$ such that those sets are invariant under $u = Fz$, $z^T = (q, \dot{q})$. In other words, such control cannot keep q inside the set.) They may be open-loop controlled holdable, however. Therefore, it is expected that in general viability will be difficult to satisfy with time-invariant closed-loop inputs (even on finite time intervals). At the same time, this study shows that results on dead-beat open-loop controllers for LTI systems [29, Theorem 5, p. 3], do not straightforwardly extend to the unilaterally constrained case; see [37] for counterexamples.
- The relationships with Bühler–Koditschek's mapping and mirror law [6] are explicited in [37].

III. The Control of Complementary-Slackness Jugglers

In this section, we present the control strategy and the closed-loop analysis for the two DOF juggler (as depicted in Fig. 1). In parallel, we propose a general analysis and control design method for the class of systems as in (11)–(13). The two DOF juggler is thus shown to constitute a simple case of such nonsmooth systems, which does not satisfy all of the desired requirements of the general framework. Moreover, it proves very useful in highlighting some peculiarities of the control design that are difficult to consider in a too-general approach, like the influence of the restitution rule and of h on property 3). One objective to be considered is to design a torque input u such that, given arbitrary initial conditions, the surface 1) hits the object with a desired preimpact angular velocity, 2) at a desired angular position (respecting the natural trajectories of the ball: the ballistic trajectory of the system imposes a time constraint on the control problem), and 3) the viability conditions hold (u is a *viable* control). In the sequel, we shall first assume that a suitable control input u satisfying 1)–3) has been designed, and we shall investigate which trajectories the object can be made to follow when it is controlled only through the impacts with the rotating table. Then, it is shown how such an input can be calculated.

A. An Intermediate Controllability Property

1) The Two DOF Juggler Case: This paragraph is devoted to investigate Property 3) in Section I-B, i.e., what we have called the controllability of the subsystem in (11) via the impacts with the restriction $m = 1$. What follows is not to be confused with the three-step recursive design method presented in Remark 2. We assume that an ideal control law, which guarantees 1)–3) above to be satisfied, exists. Hence, rather than providing us with a control law, this step aims at examining whether the impact Poincaré mapping associated with (11) is controllable when the preimpact velocity of the robot is considered as the input. Clearly, if the answer is negative, whatever the controller u we may find, the goal of the juggling (or manipulating-with-impacts) task will be limited.

To begin with, we consider the system in (5)–(10) with $\alpha = 0$ and $Y \equiv 0$ (i.e., $m = n_u = 1, n_2 = 2$). Let us denote $y_\Sigma(k)^T = (x(t_k),\, y(t_k),\, \dot{x}(t_k^-),\, \dot{y}(t_k^-))$. Integrating (5)–(7) on $(t_k,\, t_{k+1})$ and using the restitution rule in (9) and (10), it is possible to show that

$$y_\Sigma(k+1) = A_\Sigma(y_\Sigma(k),\, \Delta_k) y_\Sigma(k) + B_\Sigma(y_\Sigma(k),\, \Delta_k)\dot{\theta}(k) + C_\Sigma(\Delta_k)\Delta_k \quad (32)$$

with $\dot{\theta}(k) \triangleq \dot{\theta}(t_k^-)$ (in the following, k will stand for preimpact values). Now, if we are able to design a torque input u such that 1)–3) above are achieved, then $\dot{\theta}(k)$ can be considered as the input of the system (5), (6), i.e., (32). Indeed, notice that 2) fixes the next impact time. [In case the ball has a vertical motion, it can pass twice at the same position while going upward or downward. The choice of the desired flight-time $\Delta_k^* (= d_k)$[2] eliminates one of the two; see (22) for the one DOF juggler.] Now notice that because we assume that we know u such that $x(k+1) = x^*(k+1), y(k+1) = y^*(k+1)$ [then $\theta(k+1) = \theta^*(k+1)$ from (9)], we can express $\Delta_k = \Delta_k^*(y_\Sigma(k),\, \dot{\theta}(k),\, x^*(k+1),\, y^*(k+1))$ as

$$\Delta_k^* = \frac{x^*(k+1) - x(t_k)}{\varepsilon_{11}\dot{x}(t_k^-) + \varepsilon_{12}\dot{y}(t_k^-) + \varepsilon_{13}\dot{\theta}(k)} = \frac{\dot{y}(t_k^+) + \sqrt{\dot{y}(t_k^+)^2 - 2g(y^*(k+1) - y(t_k))}}{g}. \quad (33)$$

$\mathcal{E} = (\varepsilon_{ij})_{1\le i\le 3,\, 1\le j\le 3}$ is the state-dependent restitution matrix $(\dot{q}(t_k^+) = \mathcal{E}\dot{q}(t_k^-))$ and with $\dot{y}(t_k^+) = \varepsilon_{21}\dot{x}(t_k^-) + \varepsilon_{22}\dot{y}(t_k^-) + \varepsilon_{23}\dot{\theta}(k)$, and using the object's dynamics only (the robot dynamics and the unilateral constraint are not needed at this stage). Clearly, x^* and y^* have to be chosen on the object's trajectory. Substituting this expression into (32) yields the desired form of the partial Poincaré mapping $x_\Sigma(k+1) = F_\Sigma(x_\Sigma(k),\, \dot{\theta}(k),\, k)$ [the total Poincaré mapping state being $(y_\Sigma,\, \dot{\theta}(k),\, t_k)$] with fictitious input $\dot{\theta}(k)$

$$x_\Sigma(k+1) \triangleq \begin{bmatrix} \dot{x}(k+1) \\ \dot{y}(k+1) \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \begin{bmatrix} \dot{x}(k) \\ \dot{y}(k) \end{bmatrix} + \begin{bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix} \dot{\theta}(k) - \begin{bmatrix} 0 \\ g\Delta_k^*(\dot{\theta}(k),\, x_\Sigma(k)) \end{bmatrix} \quad (34)$$

where $\varepsilon_{11} = 1$, $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{21} = \varepsilon_{23} = 0$, $\varepsilon_{22} = -e$ if $x(k) = y(k) = \dot{x}(k) = \theta(k) = 0$. More generally, $\varepsilon_{ij} = \varepsilon_{ij}(x(k),\, y(k),\, \theta(k)) = \varepsilon_{ij}(x(k),\, y(k))$ from (9). Recall that, from the assumptions we made, the coefficients ε_{ij} depend on $x^*(k)$ and $y^*(k)$. Notice that, if the ball is initially at rest at the origin, then from (34) and (33) it follows that $\dot{x}(k+1) = \dot{y}(k+1) = \Delta_k = 0$ because one must choose $y^*(k+1) = 0$ in this case. Therefore, no impact occurs and the ball may start to slide along the bar. In other words, those initial data are such that the closed-loop impact Poincaré mapping is not defined because the vertical orbit passing at the origin cannot be controlled. It suffices, however, to make the ball leave this position to make it detach from the bar, using a suitable input. (For instance, in the one DOF case, the proposed input assures detachment even if the two bodies are initially in contact at rest or if $e = 0$ [37]. When $\lambda \le 0$, detachment becomes possible and is classically checked by searching for the first derivative $h^{(j)}$ that is >0, with $h^{(k)} = 0$, $\forall\, k < j$.) Now, derive the same partial impact Poincaré mapping when $n_u = 2$. We get

$$\begin{bmatrix} \dot{x}(k+1) \\ \dot{y}(k+1) \end{bmatrix} = \mathcal{E}' \begin{bmatrix} \dot{x}(k) \\ \dot{y}(k) \end{bmatrix} + \mathcal{E}'' \begin{bmatrix} \dot{\theta}(k) \\ \dot{Y}(k) \end{bmatrix} - \begin{bmatrix} 0 \\ g\Delta_k^*(\dot{\theta}(k),\, \dot{Y}(k),\, x_\Sigma(k)) \end{bmatrix} \quad (35)$$

for some matrices $\mathcal{E}'$ and $\mathcal{E}''$. Intuitively, the controllability and stabilizability properties of the mapping in (35) should be better than those of the mapping in (34).

2) Generalization: The interest for considering the controllability and stabilizability properties of the nonlinear discrete-time systems in (34) or (35) is that this may provide us with a sequence of "inputs" $\dot{\theta}(k)$ and $\dot{Y}(k)$. More generally, we may apply this philosophy to subsystems (11) and (12) assuming as in 1)–3) that the following occurs.

Assumption A: A viable u exists such that, given $z_1(t)$, $z_2(t)$, an impact time $t_k > t$ exists such that

- $\dot{q}_2(t_k^-)$ can be given an arbitrary value;
- $q_1(t_k)$ can be chosen as desired on the object's orbit;

Mimicking the developments for the two DOF juggler, such an assumption allows us to derive a partial impact Poincaré mapping with state $\dot{q}_1(k)$ and input $\dot{q}_2(t_k^-)$, similar to the one in (34). Recall that the viability of u guarantees that the sequence $\{t_k\}$ is countable; see Definitions 1 and 2. Also, recall that the impact Poincaré mapping is well defined. We can now set a first definition of Property 3), which we shall refine in Section III-C.

Definition 3: The subsystem in (11) is *controllable through the impacts* if its partial impact Poincaré mapping obtained

[2] The notation Δ_k^* emphasizes that the underlying philosophy is to reach a position on the object's orbit, and d_k is an *a priori* choice of Δ_k from which the attained position results.

from Assumption A, with state vector $\dot{q}_1(t_k^-)$, and considering $\dot{q}_2(t_k^-)$ as an input restricted to values satisfying $\nabla h(q(t_k), t_k)^T \dot{q}(t_k^-) < 0$, is controllable.

The restriction on the inputs is a consequence of the physics of impact (and is equivalent to $\lambda \geq 0$). Clearly, Property 1) (after Definition 2 in Section I) is crucial for the calculation of the impact Poincaré mapping. Notice that the obtained partial impact Poincaré mappings are generally strongly nonlinear in the state and the input. Various types of auxiliary feedbacks may then be applied, depending on their controllability and stabilizability properties (local stabilization, input-to-state linearization, I/O decoupling; see [22, Section 14.3]). Denote the object and the robot's inertia matrices as M_1 and M_2, respectively, (the arguments are dropped for convenience here). Define the matrices $\mathcal{M}_1 = M_1^{-1}(\nabla_{q_1} h)\, \overline{M}(\nabla_{q_1} h)^T \in \mathbb{R}^{(n_1/2)\times(n_1/2)}$, $\overline{M} = (\overline{M}_1^{-1} + \overline{M}_2^{-1})^{-1}$, $M_i^{-1} = (\nabla_{q_i} h)^T M_i^{-1} (\nabla_{q_i} h) \in \mathbb{R}^{m\times m}$, $i = 1, 2$. Then, using the restitution rule, the algebraic shock dynamics (that relate the velocity jump to the percussion vector; see, e.g., [40]), and integrating the object's motion on $[t_k, t_{k+1})$ (let us recall that the notation k in Lemma 3 means t_k^-), it is possible to show the following lemma.[3]

Lemma 3: Let the subsystem in (11) have the form $M_1 \ddot{q}_1 = \boldsymbol{g}$, where $M_1 \in \mathbb{R}^{(n_1/2)\times(n_1/2)}$ is a constant inertia matrix and $\boldsymbol{g} \in \mathbb{R}^{n_1/2}$ is a constant generalized gravity vector. Then

$$\begin{aligned}\dot{q}_1(k+1) = {}& A^*(k)\dot{q}_1(k) + B^*(k)\dot{q}_2(k)\\ &+ M_1^{-1}\boldsymbol{g}\Delta_k^*(\dot{q}_2(k), z_1(k), z_1^*(k+1))\end{aligned} \quad (36)$$

where $A^*(k) = -(1+e)\mathcal{M}_1 + I_{n_1}$ and $B^*(k) = -(1+e)M_1^{-1}\left(\nabla_{q_1} h \overline{M} \nabla_{q_2} h^T\right)$. The "object" is controllable through the impacts in the sense of Definition 3 if the nonlinear discrete-time system in (36) is controllable, under the input constraint $\nabla h(q(k))^T \dot{q}(k) < 0$. Now, assume that $\nabla_{q_1} h \overline{M} \nabla_{q_2} h^T$ has full row rank. Then, this controllability property holds if the pair $(A^*(k), B^*(k))$ is controllable and if the algebraic equation

$$v(k) = \dot{q}_2(k) + (B^*(k))^{\dagger} M_1^{-1}\boldsymbol{g}\Delta_k^*(\dot{q}_2(k), z_1(k)) \quad (37)$$

is solvable in $\dot{q}_2(k)$, with $\nabla h(q(k))^T \dot{q}(k) < 0$.

Notice that A^* and B^* depend on the ballistic constraints, i.e., $q_1^*(k)$ cannot be chosen arbitrarily; see (26)–(28). The upperscript $*$ is to emphasize that A^* and B^* depend on the desired position trajectories. In other words, the controllability Property 3) may depend on the initial data. $(B^*)^{\dagger} = B^{*T}(B^* B^{*T})^{-1} \in \mathbb{R}^{(n_2/2)\times(n_1/2)}$ denotes the Moore–Penrose generalized inverse. The proof for the first part uses standard calculations of shock dynamics. The second part follows by rewriting the system in (36) and using $v(k)$ as an intermediate input so that $\dot{q}_1(k+1) = A^*(k)\dot{q}_1(k) + B^*(k)v(k)$. From the rank condition of $B^*(k)$, the result follows. This criterion, although restricted to a specific class of "objects," is interesting because it allows us to study the influence of $h(q, t)$ on such controllability properties. In particular, because the rank condition implies $n_1 \leq n_2$, then invertibility of B^* implies $2m \geq n_1$ so that $n_1 \leq 2$, indicating that such sufficient criterion is certainly too strong (it does not apply to the two DOF jugglers with $\alpha = 0$) and must be refined. As we shall see in Section III-C, the case $m > 1$ requires particular care.

The outlined method to study controllability of the object through the impacts is expected to yield a general method of design of manipulators such that, given some prespecified goals (in terms of the object shape and motion), we may be able to design the mechanical structure of the robot in accordance. For instance it is clear that input–output decoupling of (34) with $\dot{x}(k)$ and $\dot{y}(k)$ as outputs is not possible because only one input [22] exists; hence, the possible interest to consider is $n_u = 2$ and (35), which indicates that we should preferably have $n_1 = n_2$ in general, although this is not sufficient to assure controllability of the partial Poincaré mapping F_{Σ}.

Remark 4: Bühler *et al.* [6] made similar assumptions as 1) and 2) and studied some controllability properties of a two DOF juggler performing a vertical one-juggle task. In [7] and [8], the control of a kangaroo hopping robot has been investigated. In particular, controllability properties of the impact Poincaré mapping with respect to the true inputs have been derived. In those works, the goal is to stabilize (locally) the system around a lossless natural (with no input) periodic trajectory of the system (such trajectory does not exist in all juggling systems). Also notice that viability conditions do not appear because the control is applied during the contact phase, whose length is greater than zero because it is assumed in [1], [7], and [8] that $e = 0$. Finally, notice that the underlying idea of extracting a discrete system and a controlling one variable with another coordinate as the input has been used in the control of hopping robots [1], [8], [25].

Remark 5:

- The results in [3] are not applicable (at most, they would provide us with necessary conditions) because the "input" p_k at impacts is signed, which is the reason why we have introduced the "robot" preimpact velocity through the restitution rule.
- The controllability properties of the "object" may also be analyzed using a similar basic idea to what has been done in the vibro-impact literature to prove the existence of periodic trajectories in simple impacting devices with complex dynamical behavior; see [20], [28], [36], and [40, Section 7.1.4]. This process has been advocated in [38] and will be further developed in future works.

B. Coordinate Transformations for Controller Design

The foregoing subsection has been devoted to studing systematic methods to provide a designer with sequences of a "robot's" preimpact velocities. We now pass to the second step of the design. We have solved the one DOF juggling control problem using dead-beat control laws. A first question is: is it possible to extend this kind of strategy to the two DOF case with $\alpha = 0$ and $Y \equiv 0$? In other words, can we assure the requirements 1)–3) using the ideas developed for the one DOF juggler? The answer is, in some manner, yes.

[3]We keep the notation m, although $m = 1$ in this section, but some of the developments that follow will be used in the multiple constraints case. Notice that necessarily here the codimension m_c of the striked surface satisfies $m_c = m = 1$, but for $m_c \geq 2$, $\overline{M} \in \mathbb{R}^{m_c \times m_c}$.

1) A General Framework for the SISCO[4] Case: In order to extend the previous ideas to more general complementary-slackness jugglers, let us first make the following hypotheses.

Assumption B: $n_u = 1 (= m)$.

Assumption C: The relative degrees r^*, r_λ are defined everywhere.

Assumption D: $r^* = n_2$.

Lemma 4: If Assumptions B–D are satisfied, then the system in (11)–(13) can be transformed by a local diffeomorphism $\mathcal{F}$: $z \mapsto \binom{z_1}{\mathbf{\Phi}(z_1, z_2)}$ into

$$(\Sigma)\begin{cases}\dot{z}_1 = f_1(z_1, t, \lambda)\\ \varphi^{(r^*)} = H\left(z_1, z_2, \lambda, \cdots, \lambda^{(\beta)}\right) + G(z_1, z_2)u\\ \varphi \geq 0, \quad \varphi\lambda = 0, \quad \lambda \geq 0\\ \dot{\varphi}(t_k^+) = -e\dot{\varphi}(t_k^-)\end{cases} \tag{38}$$

where

$$\beta = \begin{cases} r^* - r^\lambda, & \text{if } r_\lambda \leq r^*\\ \text{is not defined}, & \text{if } r_\lambda > r^*\end{cases}$$

$\Phi = (\varphi, \dot{\varphi}, \cdots, \varphi^{(r^*-1)})^T \triangleq (\varphi_1, \cdots, \varphi_{r^*})^T = \mathbf{\Phi}(z_1, z_2)$, and $\varphi = h$.

The proof of Lemma 4 uses standard results of nonlinear control theory and may be seen as a partial linearization. Notice that

$$D\mathcal{F} = \begin{pmatrix} I_{n_1} & 0\\ \dfrac{\partial \mathbf{\Phi}}{\partial z_1} & \dfrac{\partial \mathbf{\Phi}}{\partial z_2}\end{pmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$$

is full-rank if and only if $\partial\mathbf{\Phi}/\partial \mathbf{z_2} \in \mathbb{R}^{n_2\times n_2}$ is full-rank (which is true because Assumptions C and D; [22]; actually, setting $\lambda \equiv 0$ and $\varphi \equiv 0$, we see that $\dot{z}_1 = f_1(z_1, t)$ represents the uncontrollable zero dynamics of the system with output φ; recall [22, pp. 336–337] that the uncontrollable modes are necessarily modes of the zero dynamics). In the sequel of this subsection, we shall assume that the form (Σ) is defined globally. Write the vector fields f_1 and f_2 as $f_{10}(z_1) + \nabla_{z_1} h\lambda$ and $f_{20}(z_2) + g_{20}(z_2)u + \nabla_{z_2} h\lambda$, respectively. Denote $f^T = (f_{10}^T, f_{20}^T)$ and $g^T = (0_{n_1}^T, g_{20}^T)$. Then, if $r^\lambda < r^*$, we have

$$\begin{aligned} h^{(r^*)} &= L_f^{r^*} h + L_g L_f^{r^*-1} hu\\ &\quad \underbrace{+ L_{\nabla_z h} L_f^{r^*-1} h\lambda + \sum_{k=0}^{r^*-r^\lambda} \frac{d^k}{dt^k}[L_{\nabla_z h} L_f^{r^\lambda} h\lambda]}_{=0 \text{ if } h>0}. \end{aligned} \tag{39}$$

The other expressions for derivatives of h may be derived similarly and allow us to obtain $H(z_1, z_2, \lambda, \cdots, \lambda^{(\beta)})$ and $G(z_1, z_2) = L_g L_f^{r^*-1} h$.

Therefore, such a transformation generally yields measure differential equations with singular distributions of order ≥ 1. Notice that higher order derivatives $\varphi^{(j)}, j \geq 2$, may not be continuous, but in general are singular distributions of order $j-1$. In fact, we have the following lemma.

Lemma 5: If the system in (11) and (12) is a mechanical system with a unilateral constraint $h = \varphi \geq 0$, then $\beta = r^* - 2$.

The proof follows from the fact that, necessarily, $\dot{\varphi} = \dot{h}$ is a right-continuous function of bounded variation, and $\ddot{\varphi} = \ddot{h}$ is a measure [21]. Hence, $\varphi^{(j)}$, $j \geq 3$, is a distribution of order $j-2$, and it follows that $\beta = r^* - 2$. This result can be proved also directly, noticing that $L_{\nabla_z h} L_f h = \nabla_q h^T M^{-1} \nabla_q h$ with $M = \mathrm{diag}(\mathrm{M}_1, \mathrm{M}_2) \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$, that is different from zero as long as $\nabla_q h \neq 0$ (true by assumption), which can be checked on examples. For the two DOF juggler and the kangaroo in (40), $r^* = 2$ and $\beta = 0$. r^* is not defined globally in these cases, however, because of h. Note that, when $\varphi \equiv 0$ (the permanently constrained mode), the order of the system (Σ) is $n_1 + \beta = n_1 + r^* - 2 = 2((n_1)/(2 + n_2/2) - 1)$.

The following lemma is also true.

Lemma 6: Consider the system in (11) and (12) with output h in (13). Then, $r^* \leq n_2$.

To prove Lemma 6, let us consider $\lambda \equiv 0$. Assume that $r^* = n_2 + j$, $j \geq 1$. Then, the system can be transformed in the normal form, i.e., a chain of $n_2 + j$ integrators. Thus, the strong accessibility distribution (i.e., the controllable subspace) has everywhere dimension $n_2 + j$ [22, Th. 3.4.9, Def. 11.15], which is a contradiction because the uncontrollable subspace has dimension n_1 [the subsystem (11) with $\lambda \equiv 0$]. This process shows that the partial linearization performed in (Σ) is in some sense the best result we may expect.

Example 1: Consider now the system in Fig. 2, which represents a one DOF flexible joint robot that collides a dynamical environment. The canonical form of the dynamics as in (38) yields $\varphi^{(4)} = w(q_1, q_{21}, q_{22}, u, \lambda, \ddot{\lambda})$ because $r^* = 4$. Notice that Definition 2 may not be always appropriate in this case, because it means that the aim of the task is to control the environment, not the robot. The dynamics of the two carts with a hook yields $m = 2$, $m_c = 1$, $r_1^* = r_2^* = r_1^\lambda = r_2^\lambda = 2$. The four DOF hopper studied in [1] also fits within our framework, with $n_u = 2$, $n_1 = n_2 = 4$, and $m = 1$.

Finally, consider the kangaroo hopper in Fig. 2. It is assumed that the total mass is concentrated at the joint. Its dynamics in coordinates $(x, y, \theta_1, \theta_3)$ with $\theta_3 = I_1\theta_1 + I_2\theta_2$ is given by

$$\begin{aligned} &m_1\ddot{x} = 0, \quad m_1\ddot{y} = -m_1 g + \lambda, \quad I_1\ddot{\theta}_1 = u + l_1\lambda\cos\theta_1\\ &\ddot{\theta}_3 = l_1\lambda\cos\theta_1,\ y + l_1\sin\theta_1 \geq 0, \quad (y + l_1\sin\theta_1)\lambda = 0,\\ &\lambda \geq 0. \end{aligned} \tag{40}$$

This process is a simplified model of a kangaroo hopper, as in [1] and [7]. In (40), we may take $q_1^T = (x, y)$ and $q_2^T = (\theta_1, \theta_3)$. As expected, the fact that the constraint is frictionless hampers us to create a horizontal motion, contrary to what is proposed in [1], [7] and [8]. Hence, the system in (36) is not controllable. It is not clear at this stage how h, n_u or friction at shocks may influence the property in Definition 3.

Three-Steps Recursive Method: Let us now apply the three-step recursive control design method to the system in (38).

- *Step 1:* Choose $\dot{q}_1^*(t_{k+1}^+)$, $q_1^*(t_{k+1})$, and $q_2^*(t_{k+1})$ via $h(q_1^*(t_{k+1}), q_2^*(t_{k+1}), t) = 0$. Obtain the expressions
$$\begin{aligned} q_1^*(t_{k+1}) &= Q_1(\Delta_k, t_k, z_1(t_k^+))\\ \dot{q}_1^*(t_{k+1}^+) &= R[z_1(t_{k+1}^-), \Phi(t_{k+1}^-)]. \end{aligned} \tag{41}$$
Deduce the fictitious inputs d_k and $\Phi^*(t_k^-)$.
- *Step 2:* Choose $u = (1/G(z_1, z_2))[v - H(z_1, z_2, 0, \cdots, 0)]$ [recall the system is considered on intervals (t_k, t_{k+1})]. Choose $v = \sum_{i=0}^{p} A_{k,i}\Delta^i$.

[4]Single-input-single-constraint.

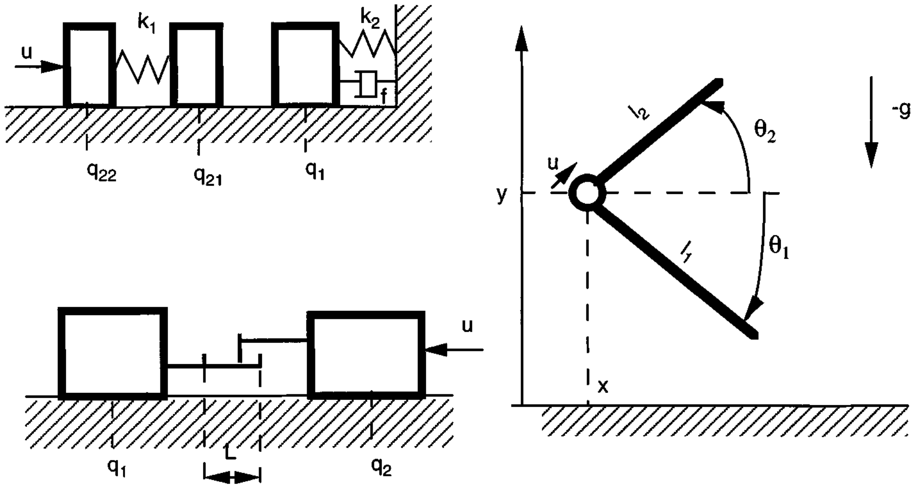

Fig. 2. One DOF flexible joint, two carts with a hook, and kangaroo robots.

Then, integration of the chain of integrators yields for $1 \le j \le r^*$

$$\varphi_j(t_{k+1}^-) = \sum_{i=j}^{r^*} \frac{\varphi_i(t_k^+)}{(i-j)!}\,\Delta_k^{i-j} + \sum_{l=0}^{p} \frac{A_{k,l}}{(l+1)\cdots(l+r^*-j+1)} \cdot \Delta_k^{l+r^*-j+1}. \tag{42}$$

Calculate the $A_{k,i}$'s by setting $\varphi_j(t_k^-) = \varphi_j^*(k)$ and $\Delta_k = d_k$. Notice that the r^* equalities in (42) can be rewritten as $\Phi = \mathcal{A}A_k + \mathcal{B}$, where $A_k^T = [A_{k,1}, \cdots, A_{k,p}]$ and $\mathcal{A} \in \mathbb{R}^{r^* \times p}$, $\mathcal{B} \in \mathbb{R}^{r^*}$. Hence, we must have $p \ge r^*$ to calculate the coefficients $A_{k,l}$.

- *Step 3:* Check viability on (t_k, t_{k+1}) with the sign of $h(\Delta) = \varphi_1(t) = \varphi(t)$ that is a polynomial in Δ, of order $(p + r^* - 2)$. For instance, for the one DOF juggler with the control as in Lemma 1, we get $h(\Delta) = \varphi(\Delta) = \Delta(\Delta - d_k)Y(\Delta)$, where $Y(\Delta) = (A_k/6)\Delta - (\dot\varphi_k/d_k)$, and we can prove that $h(\Delta) > 0$ on $(0, d_k)$ [37].

The procedure applies to the one DOF juggler defining $\varphi = y_1 - y_2$. We retrieve the same controller as in Section II, Remark 2. In general, step 3, which is crucial for satisfing the controllability property in Definition 2, will be difficult to check. It is possible, however, to numerically search for desired motions $z_1^*(k)$ such that $h(\Delta) > 0$ and to restrict the task to such motions. Moreover, notice that $h(\Delta) = \Delta(\Delta - d_k)\overline{h}(\Delta)$ for some $\overline{h}(\Delta)$, which (slightly) simplifies the problem. Most importantly, other types of controllers $v(t)$ can be tried, e.g., $v(t) = \sum_{i=0}^{p} A_{k,i}\delta_{t_i}^{(i)}$. By a density argument, we know that such an input can be approximated by smooth controllers. Also, piece wise constant inputs $v(t)$ may be applied. Notice that combinations of such controllers is also possible; see [16]. Still, another solution is to apply a control during the phases $\varphi \equiv 0$[1], [7]. As noted in Remark 4, however, this is feasible only when $e = 0$, which may be a restrictive assumption. The goal is then to control the reduced-order system $\dot z_1 = f_1(z_1, t, \lambda)$, $H(z_1, z_2, \lambda, \cdots, \lambda^{(\beta)}) + G(z_1, z_2)u_c = 0$ to get detachment at some $\tau > 0$ and with $z_1^*(t_k + \tau)$. Finally, it is possible to derive tracking controllers for the robot, by designing desired trajectories $z_{2d}(t)$ on $(t_k, t_k + d_k)$ under the constraint $h(q_1(t), q_{2d}(t), t) > 0$ and the boundary conditions $z_{2d}(t_k^+) = z_2(t_k^+)$, $z_{2d}(t_{k+1}^-) = z_2^*(t_{k+1}^-)$, $z_1(t_k^+)$, $z_1^*(k+1)$. This process may not be easy in certain cases.

Remark 6: Notice that from Lemma 3 we may deduce a sequence of inputs $\dot q_2(k) = \dot q_2^*(k)$, hence Δ_k^*. Those values can be used in Step 2. •

The explicit derivation of Q_1 in (41) is in general possible only if the vector field f_1 is linear in z_1 and is related to Property 1). This feature is common in such manipulations in which integration is needed [7], [8]. The second equation, however, can be obtained in a more general setting for mechanical systems. Indeed, if $M_1(q_1)$ and $M_2(q_2)$ denote the inertia matrices of both subsystems in (11) and (12), respectively, and $\lambda = p_k\delta_{t_k}$ at impacts, then it is easy to show that

$$p_k = -(1+e)\overline{M}(\nabla_q h)^T \dot q(t_k^-) \ge 0 \tag{43}$$

and

$$\dot q_1(t_k^+) = -(1+e)M_1^{-1}(\nabla_{q_1} h)\overline{M}(\nabla_q h)^T \dot q(t_k^-) + \dot q_1(t_k^-) \tag{44}$$

with $\overline{M}$ as in Lemma 3. Clearly, $\overline{M}$, $\overline{M}_1$, and $\overline{M}_2$ are full-rank provided $\nabla_{q_1} h$ and $\nabla_{q_2} h$ are. Thus, we have[5] the following lemma.

Lemma 7: The second algebraic equation in (41) is solvable if and only if the matrix $(\nabla_{q_1} h)\overline{M}(\nabla_{q_2} h)^T \in \mathbb{R}^{(n_1/2)\times(n_2/2)}$ has full row rank. Then

$$\dot q_2^*(t_k^-) = (\nabla_{q_1} h \overline{M} \nabla_{q_2} h^T)^\dagger \frac{M_1}{1+e} \cdot [\dot q_1^*(k) + \{(1+e)\mathcal{M}_1 - I\}\dot q_1(t_k^-)] + \eta \tag{45}$$

with $\mathcal{M}_1$, as in Lemma 3, $\Phi^*(t_k^-) = \mathbf{\Phi}(z_1^*(t_k^-), z_2^*(t_k^-))$, and $\eta \in \mathrm{Ker}[\nabla_{q_2} h^T]$.

Notice in particular that the conditions of Lemma 7 imply $n_1 \le n_2$ and $2m \ge n_1$ because $\mathrm{rank}(\nabla_{q_1} h \overline{M} \nabla_{q_2} h^T) \le \min(n_1/2, n_2/2, m)$, which indicates that in case we deal with Lagrangian mechanical systems, $m = 1$ implies $n_1 \le 2$ to fit

[5]Similarly as for Lemma 3, we keep the notation m, although only the case $m = 1$ is analyzed for the moment.

within Lemma 7 (and part 2 of Lemma 3) framework. If $m = 1$ and $n_1 > 2$ we have to search for suitable $\dot{q}_1^*$ such that $(M_1/1+e)[\dot{q}_1^*(k) + \{(1+e)\mathcal{M}_1 - I\}\dot{q}_1(t_k^-)] \in \mathrm{Im}(\nabla_{q_1} h\overline{M}\nabla_{q_2} h^T)$. Notice that the results of Lemmas 3 and 7 are not equivalent, although the well-conditioning of the matrix $\nabla_{q_1} h\overline{M}\nabla_{q_2} h^T$ is useful in both cases.

The canonical form (Σ) is not needed to perform Step 1, as the result of Lemma 7 shows. It greatly facilitates Steps 2 and 3, however. Let us notice that contrary to the controllability of the impact Poincaré mapping associated with (11), the input/output strong decoupling of the "free-motion" system with input u and output h is a property that facilitates the control design, but it is not a fundamental property.

Remark 7: The control design method based on the canonical form in (38) applies to various systems, like some hopping robots, jugglers, and manipulators with dynamic passive environments. The system in (40) and the two DOF juggler have a r^*, however, that is not defined globally. This obstacle may be overcome by either performing some other coordinate transformations or by adding a DOF and an input, i.e., by increasing n_u. Consider, for instance, the two DOF juggler as in Fig. 2 with $\alpha = 0$ and $Y \equiv 0$ ($n_u = m = 1$). It is easy to verify that the choice of the quasicoordinate $\varphi = h(x, y, \theta)$, for h as defined by the constraint in (9) yields a relative degree r^* that is not well-defined in the neighborhood of the subspace $Z_p = \{(x, y, \theta)\colon\ y\sin\theta + x\cos\theta = 0\}$. Now, if $\alpha = 0$ ($m = 1$), $n_u = 2$, we get $\ddot{h} = -(g + (u_2/m))\cos\theta - (u_1/I_2)[x\cos\theta + (y - Y)\sin\theta] + g(x, \dot{x}, \dot{y}, \theta, \dot{\theta})$ for some smooth function g. Thus, the space in which the system loses its relative degree with respect to both u_1 and u_2 reduces to $Z_p' = \{(x, Y, y, \theta) = (x, Y, Y, \pi/2)\}$. Comparison with Z_p shows that by suitably switching between u_1 and u_2, the system may be partially linearized to the canonical form in a much larger work space. The whole analysis of the two DOF juggler will be the object of future investigations and is not done here for the sake of briefness of the paper. •

C. The MIMCO Case[6]

This subsection is devoted to extending the foregoing SISCO setting to the MIMCO case ($n_u > 1$, $m > 1$). We have seen that control of the partial impact Poincaré mapping associated with (11) is facilitated if $n_1 = n_2$, even if $m = 1$ (consider, for instance, the two DOF juggler with $\alpha = 0$). The first step of the recursive method requires also in general $n_1 \leq n_2$; see Lemma 7. Moreover, it is clear that the partial I/O strong decoupling performed in (38) requires $m = n_u$. Notice that simple examples exist, in which several unilateral constraints naturally appear: hopping robots [consider, for instance, the kangaroo in (40), and rotate it to obtain a compass gait with $m = 2$], the two DOF juggler with $\alpha > 0$, building models relying on the simple rocking model [40], vibratory feeders [34], and nonprehensile manipulation systems.

1) Simple Multiple Impacts: When $m > 1$, the design of the control scheme requires more care than for the codimension one case. As pointed out above, multiple shocks create modeling problems. For control purposes, however, we can suppose that a correct impact rule has been defined, without explicitly specifying it. This process may be a solution for the control of "objects" with $(n_1/2) > 1$; see Lemmas 3 and 7. A first approach is the direct extension of the developments in Section III-A to the case of *simple multiple shocks*: in other words, the object always strikes the robot at the same singularity of the admissible domain of the configuration space [that is defined by $h_s = \cap_{1\leq i\leq m_c} h_i(q, t) = 0$, $m_c \leq m$]. Therefore, the codimension one setting extends to this codimension $m_c > 1$ setting once an impact rule is defined. To simplify, assume orthogonality of the attained constraints in the kinetic metric; i.e., $\nabla_q h_i^T M^{-1}(q)\nabla_q h_j = \nabla_{q_1} h_i^T M_1^{-1}(q)\nabla_{q_1} h_j + \nabla_{q_2} h_i^T M_2^{-1}(q)\nabla_{q_2} h_j = 0$. In this case, it is known that Newton's conjecture in (4) can be extended to the case $m_c > 1$. To simplify again, assume that $e_i = e$ for all i. Then, the settings of Lemmas 3 and 7 can be used with $m_c > 1$.

As an illustration of Lemma 3, let us consider the two DOF juggler with $\alpha > 0$ ($m_c = 2$) and $n_u = 2$, whose partial impact Poincaré mapping is given in (35). The computation of the matrix $\mathcal{E}'' = B^*(k) \in \mathbb{R}^{2\times 2}$ shows that rank $B^*(k) = 1$, using the fact that $x(t_k) = Y(t_k) - y(t_k) = 0$. Adding a horizontal DOF $X(t)$ to the robot (and modifying h_1 and h_2 in accordance), however, it is possible to show that $B^*(k) \in \mathbb{R}^{2\times 3}$ has rank 2. Thus, provided (37) is solvable, we deduce that we need $n_u = 3$, $n_2 = 6$, and $m_c = 2$ to apply Lemma 3-sufficient conditions.

We conclude that multiple shocks improve the controllability properties of the object's partial impact Poincaré mapping (see the second item of Lemma 3): this is not surprising because a multiple impact has more capabilities of reorientation of the object after a shock. Notice also that item 2) may become difficult to satisfy because it implies that the "robot" is able to strike the "object" at any point of its orbit. Because we assume that collisions are restricted to some subspace of the admissible domain boundary and because orthogonality conditions generally imply particular configurations of the whole system at t_k, the "robot" should possess enough DOF to assure 2). Consider the two DOF juggler with $\alpha > 0$ and $n_u = 2$. Obviously, the robot is not able to satisfy 2) because $m_c = 2$ and the rotational DOF $\theta(t)$ is useless in moving the point h_s in the object's configuration space (x, y). Add the horizontal DOF $X(t)$ to the robot. The table singular point h_s can now attain any point in the (x, y)-plane. Such mobility problems depend on the application at hand.

2) Successive Simple Impacts: Now, deal with tasks that consist of successive simple collisions. Indeed, it may not be desired to have $m_c \geq 2$ in certain systems, or even impossible (think of dynamic backlash: $m = 2$, but $m_c = 1$). We notice at once that the controllability as defined in Definition 3 must be modified to cope with possible successive collisions with different constraints $h_j = 0$. Indeed, the restitution rule may change from one shock to the next, hence, modifying the partial impact Poincaré mapping as in (34). In other words, the form of the mapping $x_\Sigma(k+1) = F_{\Sigma,j}(x_\Sigma(k), \dot{q}_2(k), k)$ depends on which constraint $h_j = 0$ is striked. Within this setting, we conclude that it is not possible to derive an explicit form of the application that drives the state $x_\Sigma(k)$, without taking into account the order of the attained surfaces (this is a similar

[6]Multi-input-multi-constraint.

conclusion as the one in [13] concerning modeling of multiple shocks). Assume that the vector relative degree $[r_1^*, \cdots, r_m^*]$ of the system (11), (12) with input $u \in \mathbb{R}^m$ and output $h \in \mathbb{R}^m$ is globally defined, so that strong I/O decoupling can be performed. Then, the second subsystem in (38) takes the form

$$\begin{aligned} \varphi_j^{(r_j^*)} =& H_j\left(z_1, z_2, \lambda, \cdots, \lambda^{(\beta_j)}\right) + G_j(z_1, z_2)u_j, \\ &\beta_j = r_j^* - 2, \quad y_j = \varphi_j \geq 0, \quad \varphi_j\lambda_j = 0, \\ &\lambda_j \geq 0, \ 1 \leq j \leq m \\ \dot{\varphi}_j(t_k^+) =& -e_j\dot{\varphi}_j(t_k^+) \text{ if } h_i(t_k) > 0, \\ &\text{for } i \in \{1, \cdots, m\}, \, i \neq j. \end{aligned} \tag{46}$$

We thus restrict ourselves to tasks involving a succession of simple impacts with the surfaces $\varphi_j = 0$. Let us modify Assumption A as follows.

Assumption E: u exists such that given $z_1(t)$, $z_2(t)$ an impact time $t_k^{(j)}$ with the surface $\varphi_j = 0$ exists such that

- $\dot{q}_2(t_k^{(j),-})$ can be given an arbitrary value;
- $q_1(t_k^{(j)})$ can be chosen as desired on the object's orbit;
- on $[t, t_k^{(j)})$, $h_i > 0$ for all $i \in \{1, \cdots, m\}$;
- $h_i(t_k^{(j)}) > 0$ for all $i \neq j$, $i \in \{1, \cdots, m\}$.

In other words, the "robot" can strike the "object" on any point of the object's orbit, with any preimpact velocity, with an arbitrary constraint, and with a viable control. Such viable inputs u_j can be derived similarly as in the SISCO case. Viability conditions, however, are more stringent because of the fourth item in Assumption E; see, e.g., [16] for dynamic backlash as the two carts with a hook in Fig. 2. Define a sequence $\mathcal{S}_n = \{s_1, \cdots, s_n\}$ of n indices $j \in \{1, \cdots, m\}$.[7] $\mathcal{S}_n$ therefore fixes the ordering of the successive simple impacts with $h_{s_l} = 0$. Notice that, for each n, m^n possible sequences $\mathcal{S}_n$ exist. The impact Poincaré mapping is thus defined between times $t_k^{(j),-}$ for $j \in \mathcal{S}_n$. Mimicking what we have done in (34), we get $x_\Sigma(k+1) = F_{s_l}(x_\Sigma(k), \dot{q}_2(t_k^{(s_l),-}), t_k^{(s_l)})$ on $[t_k^{(s_l)}, t_{k+1}^{(s_{l+1})})$. Then, we have on $[t_k^{(s_l)}, t_{k+2}^{(s_{l+2})})$ the application $x_\Sigma(k+2) = F_{s_{l+1}}(x_\Sigma(k+1), \dot{q}_2(t_{k+1}^{(s_{l+1}),-}), t_{k+1}^{(s_{l+1})})$, and so on for the whole sequence $\mathcal{S}_n$. Defining $j \triangleq k + jn$, $j \in \mathbb{N}$, so that j represents the number of sequences $\mathcal{S}_n$, $t_j \triangleq [t_{k+jn}, \cdots, t_{k+n+jn}]$, $U(j)^T \triangleq [\dot{q}_2(t_{k+jn}^{(s_1),-}), \cdots, \dot{q}_2(t_{k+n+jn}^{(s_n),-})]$, and $F \triangleq F_{s_n} o \cdots o F_{s_1}$: $\mathbb{R}^{n_2/2} \times \mathbb{R}^{nn_2/2} \mapsto \mathbb{R}^{n_2/2}$, we obtain

$$x_\Sigma(j+1) = F[x_\Sigma(j), U(j), t_j]. \tag{47}$$

Definition 4: Let Assumption E be true. Then, the subsystem in (11) is controllable through the impacts if at least one sequence $\mathcal{S}_n$ exists such that the associated impact Poincaré mapping in (47) is controllable with input $U(j) \in \mathbb{R}^{nn_2/2}$, whose n entries satisfy the preimpact velocity condition.

In practice, we have to fix n, then search for one sequence $\mathcal{S}_n$ among the m^n possible ones, such that controllability holds. Although it might appear clumsy, such enumerating procedure is inherent to systems involving multiple shocks [13]. If the F_{s_l}'s are nonlinear in general, only local results will be obtained. In the framework of Lemma 3, we get the special structure for F in (47)

$$\begin{aligned} \dot{q}_1(j+1) =& \mathcal{A}^*(j)\dot{q}_1(j) + \mathcal{B}^*(j)U(j) \\ &+ A_{s_n}^* \cdots A_{s_1}^* M_1^{-1}\boldsymbol{g}\Delta_{k+jn}^* \\ &+ \cdots + M_1^{-1}\boldsymbol{g}\Delta_{k+n-1+jn}^* \end{aligned} \tag{48}$$

with $\mathcal{A}^*(j) = A_{s_n}^*(k+n+jn) \cdots A_{s_1}^*(k+jn)$ and $\mathcal{B}^*$ is defined similarly. This special structure should be used in future works to prove further results on controllability and stabilizability of such closed-loop impact Poincaré mappings. The concept in Definition 4 may provide us with a starting point to study controllability (in the sense of Definition 2) of the kangaroo with $m = 2$ and successive shocks with both feet.

Remark 8—Object's Controllability: Notice that we may also consider the kangaroo in (40) and add one DOF in the leg that contacts the ground to get $n_u = 2$. It is, however possible in certain cases that the free-motion uncontrollable part of the "robot" plays a role in the overall controllability and stabilizability because it may be indirectly "controlled" via the multiplier λ, which depends on $h(q, t)$. With this in mind, we conclude that the controllability of the object (see Definition 2) is a complex property. Notice further that because of the ballistic constraints or to the designer requirements, we may have to choose $T \neq t_k$ in Definition 2 (for instance by targeting the apex of the "object's"; orbit). Then, we can state that if the object is controllable in the sense of Definition 3, if ϕ_{z_1} is controllable as in 1)[8] and if a viable input u exists, then the object is controllable in the sense of Definition 2; i.e., we can find a viable input that drives the state to the desired target at time T (perhaps after a series of impacts).

IV. Conclusions

This paper deals with the extension of feedback control strategies for one DOF jugglers previously proposed in [37] to a class of complementary-slackness "jugglers." The interest for studying the control properties of such systems is twofold: first, they belong to a large class of complementary-slackness systems whose controllability and stabilizability properties have not yet fully been understood; second, they encompass many impacting controlled systems. Although it is clear that suitable controllers will depend on the particular application at hand (the goals and the technological constraints are different from one system to another), it is important to recast the analysis of such systems into a general stabilizability and control design framework.

This study should, in our opinion, be seen as a first step toward a better understanding of control properties of the general class of complementary-slackness juggling systems as in (11)–(13). Indeed, we believe that the developments in this note pave the way toward a general control design method for such nonlinear nonsmooth systems and shed a new light on a topic that has been the object of many studies in the past 10 years.

[7] Clearly, this n has nothing to do with the number of DOF's of the system in (1).

[8] It is easily checked that this is the case for the one and two DOF jugglers, the one-DOF and the kangaroo hoppers. This process is less direct for the one DOF flexible joint robot with a passive environment as in Fig. 2. We may also imagine "objects," e.g., on moving belts, with dry friction and other nonlinear effects, for which ϕ_{z_1} takes more complex forms and is not necessarily controllable.

Two steps have been proposed. The first one is to examine the object's controllability through impacts, by studying the controllability and stabilizability properties of the closed-loop impact Poincaré mapping associated with the subsystem in (11), with the robot's preimpact velocity $\dot{q}_2(t_k^-)$ as a fictitious input. The second one concerns the controller design. It uses some state-space transformation (partial linearization), which allows us to perform a recursive algorithm from which a viable u can be calculated. Although some parts of such method have been alluded to in the related literature, their systematic analysis in a general control framework is thought to be proposed for the first time.

The limitations of the proposed design method have been pointed out, as well as some alternative paths (e.g., concerning the "robot's" control during flight phases). We have also studied the role played by some properties of the subsystems (orders, relative degrees, number of inputs) as well as the restitution rule, on the controllability of the system, both in particular cases (two DOF juggler) and in a more general framework. The two DOF juggler analysis is expected to serve as a basis for other types of systems. Future work should concern the extension of this work toward more complex complementary-slackness jugglers (adding DOF's in both the "object" and the "robot," taking into account friction and more complex restitution rules), in the outlined stabilization framework. In particular, the influence of the various characteristic numbers r^*, r_λ, n_1, n_2, m, n_u, e, m_c, the friction impulse ratio μ [4], and of the contact geometry on the intermediate controllability properties of the "object's" dynamics has been studied all through the paper and should be investigated in more detail to enlarge the class of systems to which the proposed developments apply.

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