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Passivity and Global Stabilization of Cascaded Nonlinear Systems

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***Abstract*—In this note, we present an alternative stability analysis for recent results on global stabilization of a nonlinear system in cascade with a linear system. The analysis is carried out using passivity arguments. We also present the relationship between passivity and an important class of Lyapunov functions.**

Introduction

Global stabilization of nonlinear systems has recently been studied with renewed interest [1]–[3]. The interest has been focused on partially linear composite systems due to the normal forms and zero dynamics introduced by Isidori, Byrnes, and co-workers [4], [5], [8], and [9].

In [2], the analysis has been carried out for the case of partially linear composite systems whose linear system is of relative degree one. In this case, the passivity analysis tools were used to prove global stability. On the other hand, the analysis of the general case when the linear system is of relative degree greater than one has been treated in [1]. In the latter case, global stabilization was ensured using the Lyapunov approach.

Passivity is a useful analysis tool in the sense that the results can easily be interpreted in terms of energy in the system. Besides, many important processes exhibit passivity properties as mechanical systems. Meanwhile, the composite system studied in [1] contains a linear subsystem of relative degree $r \geq 1$, and this apparently rules out passivity as an analysis technique.

In this note, we show that the results in [1] for systems with relative degree greater than one can indeed be alternatively obtained using passivity arguments. Furthermore, we present a general result which provides a deeper insight into the relationship between Lyapunov and passivity analysis tools. In particular, we show roughly that if a system is asymptotically stable in the Lyapunov sense and the Lyapunov function is the sum of two positive definite functions, then the system can be represented as the feedback interconnection of two passive nonlinear systems. Reference [3] gives an interpretation in terms of the passivity of the problem also when the first subsystem is nonlinear. One of our motivations is to extend the results in [3] to the case $r > 1$.

We will consider, as in [1], the partially linear composite system

$$\dot{x} = f(x,0) + G(x, \xi_0, \xi)\xi, \qquad x \in R^n \tag{1}$$

$$\dot{\xi}_0 = A_0\xi_0 + A_1\xi_1 \qquad \xi_0 \in R^{q_0} \tag{2a}$$

$$\dot{\xi}_1 = \xi_2 \tag{2b}$$

$$\vdots \qquad\qquad \xi_i \in R^m, i = 1,\cdots, r$$

$$\dot{\xi}_{r-1} = \xi_r \tag{2c}$$

$$\dot{\xi}_r = u_r,\ y = \xi_1 \qquad u_r \in R^m,\ y \in R^m. \tag{2d}$$

The nonlinear system in (1) is already in a particular form where G depends only on the output $y = \xi_1$ and the linear zero-dynamics ξ_0 is induced by this output. It is assumed that the equilibrium $x = 0$ of $\dot{x} = f(x,0)$ is globally asymptotically stable and a smooth Lyapunov function $V(x) > 0, x \neq 0; V(0) = 0$ is known such that $V(x) \to \infty$ as $\|x\| \to \infty$ and

$$\nabla V(x) f(x,0) < 0 \qquad \text{for all } x \neq 0.$$

The linear part of the system is in the form to which every invertible relative degree r system can be transformed using the special coordinate basis of [7]. We assume also that the zero dynamics is stable but without loss of generality, we consider that A_0 does not have an asymptotically stable part, i.e.,

$$A_0^T + A_0 = 0. \tag{3}$$

We then have the following.

Proposition 1: The composite system (1)–(3) is globally asymptotically stabilizable at $(x, \xi) = (0,0)$ by a smooth state feedback control.

The above proposition has been proved in [1] using the Lyapunov approach. In spite of the fact that the linear subsystem (2) is of relative degree $r \geq 1$, we show next that a proof of Proposition 1 can be developed using passivity arguments.

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Proof: Consider the case $r = 2$, i.e.,

$$\dot{x} = f(x,0) + G(x, \xi_0, \xi_1)\xi_1 \tag{4}$$

$$\begin{bmatrix} \dot{\xi}_0 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u_2 \tag{5}$$

$$y = C^T \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} \qquad C^T = [0 \quad I \quad 0]. \tag{6}$$

The following variable modification will allow us to obtain a positive real linear subsystem

$$\xi_2 = u_1(x, \xi_0, \xi_1) + \tilde{\xi}_2 \tag{7}$$

where u_1 is to be chosen later and $\tilde{\xi}_2$ is a new variable.

Therefore, (5) can be rewritten as

$$\begin{bmatrix} \dot{\xi}_0 \\ \dot{\xi}_1 \\ \dot{\tilde{\xi}}_2 \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \tilde{\xi}_2 \end{bmatrix} + Bu_1 + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} (u_2 - \dot{u}_1) \tag{8}$$

where

$$B = [0 \quad I \quad 0]^T. \tag{9}$$

Note that $B = C$ so that the system in (6) and (8) verifies the condition $PB = C$ with $P = I$ which is one of the conditions for the closed-loop system to be positive real. Therefore, we only need to define u_1 and u_2 in such a way that the closed-loop system matrix satisfies some stability conditions [see (16)]. For that purpose let us consider the following:

$$u_2 = \dot{u}_1 - \xi_1 - \frac{1}{2}\tilde{\xi}_2 \tag{10}$$

and

$$u_1 = -A_1^T \xi_0 - \frac{1}{2}\xi_1 + v(x, \xi_0, \xi_1). \tag{11}$$

Therefore, (6) and (8) can be rewritten as

$$\dot{\tilde{\xi}} = A\tilde{\xi} + Bv(x, \xi_0, \xi_1) \tag{12}$$

$$y = C^T \tilde{\xi}$$

where

$$\tilde{\xi}^T = [\xi_0 \quad \xi_1 \quad \tilde{\xi}_2] \tag{13}$$

$$A = \begin{bmatrix} A_0 & A_1 & 0 \\ -A_1^T & -\frac{1}{2}I & I \\ 0 & -I & -\frac{1}{2}I \end{bmatrix} \tag{14}$$

and

$$B = C \tag{15}$$

$$A + A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \leq 0. \tag{16}$$

Therefore, the system (12) is positive real. In order to obtain two passive systems connected in feedback, it suffices to define v as follows:

$$v(x, \xi_0, \xi_1) = -G^T(x, \xi_0, \xi_1)\nabla V(x). \tag{17a}$$

Define also

$$y_{NL} = -v(x, \xi_0, \xi_1). \tag{17b}$$

The nonlinear system in (4) with output defined in (17) is passive as shown next. From (4) and (17)

$$\frac{dV(x)}{dt} = \nabla V(x)^T \dot{x} = \nabla V(x)^T [f(x,0) + G(x, \xi_0, \xi_1)\xi_1]$$

$$= \nabla V(x)^T f(x,0) + y_{NL}^T \xi_1. \tag{18}$$

Then

$$\int_0^t y_{NL}^T \xi_1 \, dt = V(x(t)) - V(x(0)) - \int_0^t \nabla V(x)^T f(x,0)\, dt$$

$$> \alpha(\|x\|) - V(x(0)) \tag{19}$$

where $\alpha(\cdot)$ is a function of class K (see [6]).

The linear system in (12)–(16) is also passive since

$$\frac{1}{2}\frac{d}{dt}\tilde{\xi}^T\tilde{\xi} = \tilde{\xi}^T\dot{\tilde{\xi}} = \tilde{\xi}^T A\tilde{\xi} + \tilde{\xi}^T Bv$$

$$= \frac{1}{2}\tilde{\xi}^T(A + A^T)\tilde{\xi} + \tilde{\xi}^T Cv$$

$$= -\frac{1}{2}\left(\|\xi_1\|^2 + \|\tilde{\xi}_2\|^2\right) + y^T v \tag{20}$$

then

$$\int_0^t y^T v \, dt = \frac{1}{2}\int_0^t \left(\|\xi_1\|^2 + \|\tilde{\xi}_2\|^2\right) dt + \frac{1}{2}\|\tilde{\xi}\|^2 - \frac{1}{2}\|\tilde{\xi}(0)\|^2. \tag{21}$$

Adding (19) and (21) and taking (17) into account we obtain

$$0 \geq \frac{1}{2}\int_0^t \left(\|\xi_1\|^2 + \|\tilde{\xi}_2\|^2\right) dt + \frac{1}{2}\|\tilde{\xi}\|^2 - \frac{1}{2}\|\tilde{\xi}(0)\|^2 + \alpha(\|x\|) - V(x(0)). \tag{22}$$

Therefore, ξ_0, ξ_1, $\tilde{\xi}_2$, and x are bounded and ξ_1 and $\tilde{\xi}_2$ are L_2. This implies that $v(x, \xi_0, \xi_1)$ in (17) is also bounded and in view of (12) $\dot{\tilde{\xi}}$ is also bounded. Therefore, ξ_1 and $\tilde{\xi}_2$ converge to zero. Since all the system variables remain bounded and since the second term on the RHS of (4) converges to zero we conclude, in view of the asymptotic stability of the autonomous system $\dot{x} = f(x,0)$, that x converges to zero.

In view of the assumption on the smoothness of $V(x)$, $f(x, \xi)$, and $G(x, \xi)$ it follows from (7), (11), and (17) that ξ_2 is also bounded.

The procedure can be extended for the case $r > 2$ as was done in [1] using the Lyapunov approach.

Lyapunov Stability Implies Passivity in a Particular Case

We now present a generalization of the result given in the foregoing section. Indeed it is well known that the passivity theorem proof can be carried out using a Lyapunov approach. On the other hand, it is not clear whether Lyapunov stability implies passivity. In this section, we show that it is the case for a particular class of Lyapunov function which is very often encountered in the literature. The result is given in the following proposition and was motivated by [3] and [10].

Proposition 2: Suppose that a system is Lyapunov stable and the Lyapunov function $V(x, \xi)$, where x, ξ is the state of the system, satisfies

1) $$V(x, \xi) = V_1(\xi) + V_2(x),$$
$$V_1, V_2 \text{ positive definite functions} \quad (23)$$

2) $$\dot{V}(x, \xi) = -\lambda_1 \beta_1(\|\xi\|) - \lambda_2 \beta_2(\|x\|) \quad (24)$$

where β_1 and β_2 are class K functions and $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$. Then there exist two passive systems connected in feedback having x and ξ as states, respectively. These systems are defined as follows:

$$\dot{\xi} = F(\xi) + G(\xi, x)u \quad (25)$$

$$y = G^T(\xi, x)\frac{\partial V_1}{\partial \xi} \quad (26)$$

and

$$\dot{x} = H(x) + J(\xi, x)y \quad (27)$$

$$-u = J^T(\xi, x)\frac{\partial V_2}{\partial x} \quad (28)$$

where $G(\xi, x)$ and $J(\xi, x)$ are arbitrary functions and $F(\xi)$ and $H(x)$ are such that

$$\frac{\partial V_1^T}{\partial \xi}F(\xi) \leq 0 \quad (29)$$

and

$$\frac{\partial V_2^T}{\partial x}H(x) \leq 0. \quad (30)$$

Remarks:

1) Systems (25)–(28) satisfy sufficient conditions to be passive as will be shown later. Conditions (29) and (30) imply stability of the two systems mentioned above.
2) The systems in (25)–(28) should be interpreted as two nonlinear passive systems. The input and output of these systems should be chosen in such a way that they satisfy (26) and (28) where $G(\xi, x)$ and $J(\xi, x)$ are arbitrary functions.
3) The above proposition states that every time a system is Lyapunov stable with a Lyapunov function satisfying (23) and (24), then there exist two nonlinear passive systems connected in feedback as in (25)–(28).

Proof: Let us first prove that the two systems in (25)–(28) are passive

$$\dot{V}_1 = \frac{\partial V_1^T}{\partial \xi}\dot{\xi} = \frac{\partial V_1^T}{\partial \xi}F(\xi) + \frac{\partial V_1^T}{\partial \xi}G(\xi, x)u$$
$$= \frac{\partial V_1^T}{\partial \xi}F(\xi) + y^T u \quad (31)$$

or

$$\int_0^t u^T y = V_1(\xi(t)) - V_1(\xi(0)) - \int_0^t \frac{\partial V_1^T}{\partial \xi}F(\xi)\, dt. \quad (32)$$

Also

$$\dot{V}_2 = \frac{\partial V_2^T}{\partial x}\dot{x} = \frac{\partial V_2^T}{\partial x}H(x) + \frac{\partial V_2^T}{\partial x}J(\xi, x)y$$
$$= \frac{\partial V_2}{\partial x}H(x) - u^T y \quad (33)$$

or

$$\int_0^t (-u)^T y\, dt = V_2(x(t)) - V_2(x(0)) - \int_0^t \frac{\partial V_2^T}{\partial x}H(x)\, dt. \quad (34)$$

Therefore, the two systems in (25)–(28) are passive independently of J and G as long as F and H satisfy (29) and (30).

Adding (32) and (34), and using (31), (33), and (24) we get

$$V_1(\xi(t)) - V_1(\xi(0)) + V_2(x(t)) - V_2(x(0))$$
$$+ \int_0^t \{\lambda_1 \beta_1(\|\xi\|) + \lambda_2 \beta_2(\|x\|)\}\, dt = 0. \quad (35)$$

Thus, ξ and x are bounded. Depending on the nature of β_1 and β_2, ξ and x may be L_p bounded functions.

Conclusions

This note has pointed out the relevance of passivity in the global stabilization of nonlinear systems. It has been shown that recent results in [1] about stabilization of partially linear composite nonlinear systems using the Lyapunov approach can also be obtained via passivity arguments in spite of the fact that the linear subsystem is of relative degree greater than one. Furthermore, it has been proved that every time a system admits a Lyapunov function composed of the sum of two positive definite functions, there exists an interpretation in terms of two passive systems connected in the feedback.

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