Some results on the controllability of planar variational inequalities

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Abstract
This note deals with the controllability of a class of planar complementarity dynamical systems, which can also be viewed as planar evolution variational inequalities. It is shown that the complementarity conditions influence the controllability of the system a lot.
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1. Introduction

Hybrid dynamical systems constitute a very large class of systems [13]. It is consequently necessary to focus on specific subclasses to make their study possible, see e.g. [1] for controllability issues in piecewise-linear systems. An interesting subclass is made of the so-called complementarity systems [2,8]. Similar to the fact that the stability of unilaterally constrained systems can significantly differ from that of their unconstrained counterpart [6,11], it will be shown that their controllability properties can differ a lot as well. This reinforces the fact that such nonsmooth dynamical systems deserve full attention and are not a mere extension of unconstrained or bilaterally constrained systems. In this note, we will restrict ourselves to a narrow class of complementarity-systems, that we call planar evolution variational inequalities. These systems are also sometimes called projected dynamical systems [4,11] and are used to model the dynamics of oligopolistic markets, spatial price equilibrium, elastic demand traffic equilibrium [11]. As illustrated at the end of the note, they can also model some electrical circuits with ideal diodes. In this note it is shown that the controllability of such systems depends a lot on the convex set within which the state is constrained to evolve.

2. Planar evolution variational inequalities

The linear complementarity systems (LCS) [8] we are dealing with in this study, possess the following
The LCS in (1) is equivalent to the linear evolution
\[ \dot{z}(t) - Az(t) - Bu(t) + z(t) \geq 0, \quad \forall v \in K, \]
\[ z(t) \in K, \quad \forall t \geq 0, \]
where \( z = (z_1, z_2)^T \in \mathbb{R}^2 \), \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), \( K = (z \in \mathbb{R}^2 | C_1 z_1 + C_2 z_2 + d \geq 0) \).

The equivalence between (1) and (2) is obtained by noting that
\[
\begin{cases}
\dot{z}(t) = Az(t) + Bu(t) + C^T \dot{\lambda}, \\
0 \leq \dot{\lambda} \perp Cz(t) + d \geq 0
\end{cases}
\]
which is equivalent to the so-called projected dynamical systems [9], which is in turn equivalent to the LCS in (1) or (2) are the topic of the study. Since the studies on controllability of this type of dynamical systems are rare, this paper nevertheless has some interest. The following lemma is a direct consequence of [7, Corollary 2.2]:

**Lemma 1.** Consider the system in (1). For all continuous and locally differentiable inputs \( u(\cdot) \), a continuous and right-differentiable solution with locally bounded derivative exists and is unique on \([0, +\infty)\).

Let us now introduce a controllability definition.

**Definition 1.** The system in (1) (equivalently in (2)) is said to be \( K \)-controllable, if any state \( z_f \in K \) can be reached from any state \( z_i \in K \), in a finite or infinite time \( T \), and with an admissible input \( u(\cdot) \).

Admissibility of the input means that the well-posedness conditions of Lemma 1 are respected. We do not make the difference between finite and infinite \( T \) to simplify the presentation (as we shall see below), this allows us to consider the controllability in the whole of the closed convex set \( K \) without excluding some isolated points of the boundary \( \partial K \).

The objective of this work is to prove that, under some restrictions on the convex set \( K \), \( K \)-controllability holds. To begin with and to motivate the study, let us remark that in case \( m = 1 \) and \( K = \{ z | z_2 \geq c, c < 0 \} \), then surely the system is not \( K \)-controllable. Indeed \( z_1 \) can only move from the left to the right in the phase plane, since \( \dot{z}_1 = z_2 > c > 0 \). This controlled VI is accessible [12] with reachable subspaces from \((z_1(0), z_2(0))\) equal to \( \{ (z_1, z_2) \mid z_1 \geq z_1(0), z_2 \geq c \} \), but not \( K \)-controllable.

Let us note that adding some “imaginary” state re-initialization rules on \( \partial K \) such that \( K \)-controllability holds, is not envisaged here since the dynamical systems in (1) or (2) are the topic of the study. However motivated by this simple example of non-controllability, one guesses that a crucial step in the study will be to prove whether or not one is able to move on \( \partial K \) in order to reach some states which are otherwise unreachable. Due to the complementarity conditions (third line in (1)) which imply that the vector field is modified when \( \partial K \) is attained, this will under certain conditions be possible.
3. Main result

The following assumption is made and supposed to hold in the sequel:

**Assumption 1.** The set $K$ has a positive measure in $\mathbb{R}^2$.

It is easy to construct $C$ and $d$ in (1) such that indeed $K = \emptyset$ or it has zero measure. Polyhedra with a positive area are an example of sets $K$, as well as cones (see Fig. 1), or half-planes.

Let $C_1 = (a_1, \ldots, a_m)^T$, $C_2 = (b_1, \ldots, b_m)^T$, $d = (d_1, \ldots, d_m)^T$ and let us denote the faces of the convex set $K$ as $D_i$, such that $D_i \subset \{z|a_iz_1 + b_iz_2 + d_i = 0\}$ and $\bar{D}_i = \{z|a_iz_1 + b_iz_2 + d_i = 0\}$. In other words the faces are segments $D_i$ (possibly unbounded, like in the case $K$ is a cone, or if $K$ is defined as a half-space), and the segments can be extended to straight lines $\bar{D}_i$ whose equations in the plane are $a_iz_1 + b_iz_2 + d_i = 0$, $1 \leq i \leq m$. For instance in Fig. 1, and considering the set $K_1$, one has $D_1 = A'A$ whereas $\bar{D}_1$ is the line passing through $A'$ and $A$ and intersecting $\{z|z_2 = 0\}$ at $B$. Let us place ourselves in the phase plane of the system, with the two axes $(0, z_1)$ and $(0, z_2)$.

Then the following is true.

**Proposition 1.** The system in (1) (equivalently in (2)) is $K$-controllable, if and only if, there is no face of $K$ such that:

- there is a portion of $D^i$ with finite negative slope on the right (resp. left) of the point $\bar{D}^i \cap \{z|z_1 = 0\}$, when $K$ is below (resp. above) $D^i$;
- $D^i$ is vertical and above (resp. below) $\{z|z_2 = 0\}$ if $K$ is on the right (resp. left) of $D^i$;
- $D^i$ is horizontal and in the half-space $\{z|z_2 < 0\}$ (resp. $\{z|z_2 > 0\}$) if $K$ is below (resp. above) $D^i$;
- $D^i = \{z|z_2 = 0\}$.

For instance in Fig. 1, the faces $A'A$ of $K_1$, or $DC$ of $K_4$, preclude controllability because they satisfy the first item.

Let us state intermediate results which characterize the motion on the boundary $\partial K$ (Fig. 2). The proof of Proposition 1, will then be a direct consequence of Lemma 2. In the next lemma, we place ourselves in the case when there is a single constraint and we study the behavior of the system on this constraint. When $K$ has several faces it will suffice to consider each of them separately and apply the results of the lemma independently to each constraint. Let us consider the system in (1) or (2), with $C_1 = a \in \mathbb{R}$, $C_2 = b \in \mathbb{R}$, $d = c \in \mathbb{R}$. Let us define the coordinate change $x_1 = bz_1 - az_2 + \frac{bc}{a}$, $x_2 = az_1 + bz_2 + c$.

![Fig. 1. Examples of K-controllable and K-uncontrollable systems.](image-url)
Fig. 2. Trajectories on $\partial K$ (vertical faces).

We denote as $P$ the intersection between the line $az_1 + bz_2 + c = 0$ and the $z_1$-axis, i.e., $P$ is the origin of the new frame $(x_1, x_2)$ and the constraint boundary is $\{x_1 = 0, x_2 \geq 0\}$. The new coordinate frame $(x_1, x_2)$ is depicted in Fig. 3.

Then the following holds:

**Lemma 2.**

(i) Positive slope. If $-a/b > 0$, $b \neq 0$, then any point $x_{1f}$ on the constraint can be attained from any point $x_{1i} \leq x_{1f}$, and any point $x_{1f}$ on the constraint can be attained from any point $x_{1i} \geq x_{1f}$ only on the axis $x_1 \in (P, +\infty)$. Moreover the point $P$ can be attained from any $x_{1i} > 0$ only asymptotically.

(ii) If $a > 0$ then the boundary is a horizontal line $z_2 = -c/a$ and

- If $(a > 0$ and $c > 0)$ or $(a > 0$ and $c < 0)$, trajectories move from the left to the right.
- If $(a > 0$ and $c > 0)$ or $(a > 0$ and $c < 0)$, trajectories move from the right to the left.
- If $c = 0$ then the system remains stuck on $\partial K$ at the contacting point.

(iii) If $b = 0$ then the boundary is a vertical line $z_1 = -c/b$ and

- If $(b > 0$ and $c > 0)$ or $(b > 0$ and $c < 0)$, the system is controllable in the set $\{z_2 < 0\}$ and any trajectory initialized in the set $\{z_2 > 0\}$ detaches from $\partial K$.
- If $(b > 0$ and $c > 0)$ or $(b > 0$ and $c < 0)$, the system is controllable in the set $\{z_2 > 0\}$ and any trajectory initialized in the set $\{z_2 < 0\}$ detaches from $\partial K$.

Let us note that the case $a = b = 0$ is meaningless since the system is no longer constrained, hence it is not treated in Lemma 2. We note that the two depicted cases can be rotated to obtain the admissible domain below the boundary. The axis $(P, z_2)$ points inside the admissible set $K$. The dashed arrows on $\partial K$ indicate the directions in which trajectories can be controlled on $\partial K$. In Fig. 1 the arrows also indicate the possible directions of motion on $\partial K$. Due to the complementarity conditions, it follows that in some regions of $\partial K$, trajectories are restricted to move in a single direction (otherwise they leave $\partial K$). The cases when the boundary is vertical, is depicted in Fig. 2.

**Proof of Lemma 2.** (i) It is simple to calculate that the dynamics (1) in the coordinates $(x_1, x_2)$
We conclude that the dynamics on a face included in $\{x \mid x_0 = 0\}$ is that both $x_1$ and $x_2$ keeps moving on a face included in $\{\dot{x} \leq b(\{x \mid x_0 = 0\})\}$, where $\dot{x}$ is the new input. We notice that if $-x_1 - (b/a)v = 0$ then the system grazes $\partial K$. If $-a/b > 0$, $b \neq 0$, then necessarily $v \leq -a/b x_1$, and $v$ can be chosen $< 0$ so that $x_1$ can be made to decrease while staying on $\partial K$. If $-a/b < 0$, $b \neq 0$, then necessarily $v \geq -a/b x_1$. If $x_1 < 0$ then $v > 0$, so on $(-\infty, 0)$, $x_1$ can only increase. On $(0, +\infty) \ni x_1$, one can choose $v = -a/b x_1$ so that $P$ is attained only asymptotically from any $x_1 > 0$.

(ii) Now if $a = 0$ (and consequently $b \neq 0$), the dynamics on $\partial K$ is given by

$$\begin{align*}
\dot{x}_1(t) &= -c/b, \\
\dot{x}_2(t) &= -c/b, \\
u(t) + b\dot{x} &= 0 \quad \text{and} \quad \dot{x} \geq 0 \Rightarrow bu(t) \leq 0.
\end{align*}$$

This is obtained in a similar way as above, noting that on $\partial K$ ones has $b \dot{x}_2 = 0$ and $0 \leq \dot{x} \perp b \dot{x}_2 \geq 0$. The results follow. The detachment from the surface $b \dot{x}_2 + c = 0$ occurs, if and only if, $b \dot{x}_2(t_d) > 0$ at some time $t_d$, i.e., $bu(t_d) + b^2 \dot{x}_1(t_d) = bu(t_d) > 0$ (indeed $\dot{x}(t_d) = 0$ from the complementarity conditions).

(iii) If $b = 0$ (and consequently $a \neq 0$) the dynamics on $\partial K$ is given by

$$\begin{align*}
\dot{z}_1(t) &= -c/a, \\
\dot{z}_2(t) &= u(t), \\
z_2(t) + a \dot{z} &= 0 \quad \text{and} \quad \dot{z} \geq 0 \Rightarrow az_2(t) \leq 0.
\end{align*}$$

The results stated in Lemma 2 (iii) are a direct consequence of (10). □

**Lemma 3.** The unilateral constraint $\{x \mid x_2 = 0\}$ in (1) can be activated or deactivated with a continuous input signal $u(\cdot)$.

**Proof.** Let us consider (1) or equivalently (4). The contact phases, or active constraint, are characterized by $\dot{\lambda} \geq 0$ and $x_2 = 0$ whereas the non-contact phases, or inactive constraint, correspond to $\dot{\lambda} = 0$ and $x_2 > 0$. When steering the state inside $K$ (i.e., in $K \setminus \partial K$) it is always possible to attain the boundary $\partial K$, and to
remain on $\partial K$, with a continuous input. Indeed this amounts to finding a path in the phase plane ($z_1, z_2$), parameterized by $t$, linking two points $z^0 \in \operatorname{Int}(K)$ and $z^1 \in \partial K$, such that its second derivative with respect to $t$ satisfies the inequality in (7) on $\partial K$. Detachment can also be forced with a continuous control input. Indeed one sees from (5) that as soon as $\partial K$ is attained $\lambda$ is the solution of a linear complementarity problem (LCP) with matrix $a^2 + b^2 > 0$ (a scalar in this case) and consequently depends continuously on $u(\cdot)$ (see e.g. [5, Example 4.8.15]). One can speak of a controlled LCP in (5) which can be controlled with a continuous input. Consequently the controllability result holds with input signals $u(\cdot)$ which are continuous and piecewise differentiable. This guarantees the existence of a unique solution of (1) or (4) as a result of Lemma 1. It is noteworthy that these results still hold if the codimension of $\partial K$ is $\geq 2$ (activation or detachment at corners of $\partial K$). □

**Proof of Proposition 1.** The proof is done by observing that under the stated conditions, and from Lemmas 2 and 3, then any point in $K$ can be steered by a continuous $u(\cdot)$ to any other point in $K$. Indeed if a state $z_f$ cannot be attained from $z_l$ via a trajectory in $K \setminus \partial K$, then a portion of path can be tracked on $\partial K$. Concatenating paths in the interior of $K$ and on $\partial K$ allows one to construct a path linking $z_l$ to $z_f$. The conditions of Proposition 1 are sufficient but can also be seen to be necessary, for if one of them fails then there exist couples of states in $K$ which cannot be joined by a controlled trajectory. □

**Examples.** From the results of Lemma 2, one sees that the boundary of the domain $K_3$ in Fig. 1 can be tracked clockwise. Consequently any point $z_f$ on the right of the line $(l)$ can be attained from any point $z_l$ on the left of $(l)$. There has to be a portion of the trajectory that evolves on $\partial K_3$ to reach $z_f$ from $z_l$. Let us consider the set $K_1$ in Fig. 1. The system is not $K_1$-controllable because the only way to attain a point on the left of the vertical line $(l)$ from a point on the right of $(l)$, is to follow the boundary $\partial K_1$. However once the point $A$ has been reached, it is impossible to move on $\partial K_1$ toward $A'$. The system can be steered on the line $AA'$ only in the direction of $B$. Consequently all points of $K_1$ which are situated on the left of $(l)$, cannot be attained from points in $K_1$ on the right of $(l)$. It is noteworthy that even small-time local controllability [14] may fail. For instance two arbitrarily close states $z_i$ and $z_f$ in $K_1$, with $z_i$ on the right of $(l)$ and $z_f$ on the left of $(l)$, cannot be joined by a solution of (1) with some control $u(\cdot)$. Consider now $K_2$. Then trajectories can be controlled from $E$ to $C$, though $C$ is reachable in infinite time only. Assume that $C$ is just below the axis $\{z|z_2 = 0\}$. It follows from Lemma 2 that $\partial K_2$ can be tracked clock-wise by applying some suitable control input. Thus, the points on the right of the vertical line $(l')$ can be steered to anywhere in $K_2$ by first moving on $FE$. One may say that the dynamics is suitably modified on the boundary $FE$ so that $z_1$ can decrease in the first quadrant. In the same way the system is $K_5$-controllable, but it is not $K_4$-controllable (the states on the left of the line $(l)$ cannot be reached from the states in $K_2$). The system is $K_5$-controllable since as illustrated a state $z_f$ that cannot be attained from $z_l$ via a trajectory which remains in $K \setminus \partial K$, can be attained via a path $z_lABz_f$.

**Remark 1.** As we said after Definition 1, including infinite time $T$ in the controllability allows us to disregard some isolated points of $K$ that may not be reachable in finite time. This is the case for the domain $K_2$ where the point $C$ can be attained asymptotically only.

4. An example

Let us consider the simple electrical circuit in Fig. 4, where $R$ is a resistor, $L$ is an inductor, $C$ is a capacitor, and the diode is supposed to be ideal. Its dynamics is given by

$$
\dot{z}_1(t) = z_2(t),
\dot{z}_2(t) = -\frac{R}{L} z_2(t) + \frac{u(t)}{L} - \frac{1}{LC} z_1(t) - \frac{1}{L} \lambda,
0 \leq \lambda \leq -z_2(t) \geq 0,
\tag{11}
$$

where $z_1(\cdot)$ is the time integral of the current across the capacitor, $z_2(\cdot)$ is the current across the circuit and $-\lambda$ is the voltage of the diode, $u(\cdot)$ is a voltage control. One has $K = \{z|z_2 \leq 0\}$. One sees that this system is not controllable by simple application of Proposition 1. One may transform the system in (11) into the canonical form in (1), by applying a pre-feedback $u(z_1, z_2) = Lu(t) + (R/L)z_2 + (1/LC)z_1$. In fact the state $z_1(\cdot)$ can only decrease, or be
controlled to a constant value on \( \partial K \). Consequently the system in (11) is not \( K \)-controllable. This is intuitively sound since it corresponds to having the capacitor loaded with a non-positive current at all times.

5. Conclusion

In this note, we have proposed a characterization of the controllability properties of planar evolution variational inequalities with control input. These systems are a subclass of complementarity dynamical systems. They are nonsmooth and nonlinear. The material in this note relies heavily on the properties of the system on the boundary of the constraint set and on the behaviour of the trajectories of planar systems in their phase plane. Consequently an extension of this work should rely on the analytical tools in [3] that characterize the control capabilities of a system, on the boundary of its admissible domain. The class of systems that is considered is a narrow class of complementarity dynamical systems. However the results show that the controllability of complementarity dynamical systems differs significantly from that of unconstrained systems.

References