A non-smooth optimization method for the friction problem in computational mechanics

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1 Rigid-bodies dynamics

2 Optimization-based formulation

3 Existence, non-uniqueness

4 Numerical experiments
Mechanical problem

- simulate dynamics of *rigid* bodies
- with *unilateral constraints* and *friction*
- typical example: *granular material* + gravity
Kinematics

- \( v = (\dot{x}, \dot{y}, \dot{\theta}) \) is the generalized velocity
- \( u = (u^1, u^2) \) is the velocity at contact points
- \( r = (r^1, r^2) \) is the reaction at contact points
Discretized problem: time integration, compute contact forces

Let (second-order cone)
\[ K_\mu = \{ \|r_T\| \leq \mu r_N \} \subset \mathbb{R}^3 \]

At each contact, disjunction:

- either \( r = 0 \) and \( u_N \geq 0 \) (take off)
- or \( r \in \text{int}(K_\mu) \) and \( u = 0 \) (sticking)
- or \( r \in \partial K_\mu \setminus 0 \) and \( u_N = 0 \) with \( u_T \) opposed to \( r_T \) (friction)
Disjunction can be formulated as complementarity...

...by setting $\tilde{u} = u + \mu e \|u_T\|$: 

$$\begin{cases} 
\tilde{u} = u + \mu \|u_T\| e \\
K_\mu \ni r \perp \tilde{u} \in K^*_\mu 
\end{cases}$$  \hspace{1cm} (1)\]
Outline

1. Rigid-bodies dynamics
2. Optimization-based formulation
3. Existence, non-uniqueness
4. Numerical experiments
Altogether, we want to solve:

\[ \begin{align*}
Hr &= Mv + f \quad \text{[Newton’s law]} \quad (a) \\
u &= H^\top v + \mu Es \quad \text{[kinematics]} \quad (b) \\
L \ni r \perp u &\in L^* \quad \text{[Coulomb’s law]} \quad (c) \\
s^i &= \|u_T^i\| \quad \forall i \\
\end{align*} \]

(2)

with

- \( E = \text{Diag}(e) \)
- \( L = \prod_i K_{\mu_i} \)
- \( M \in S_n^{++} \) (mass matrix)
- no restitution for simplicity
Case $\mu = 0$

System reduces to:

\[
\begin{aligned}
H r &= M v + f \\
u &= H^T v \\
r_T &= 0, \quad r_N \geq 0 \\
u_T &\in \mathbb{R}^2, \quad u_N \geq 0 \\
r &\perp u
\end{aligned}
\]

(3)

Recognize optimality conditions of a QP! [Moreau]

\[
\begin{aligned}
\min J(r) := \frac{1}{2} r^T (H^T M^{-1} H) r - (H^T M^{-1} f)^T r \\
r_T &= 0 \\
r_N &\geq 0
\end{aligned}
\]

(4)

where $u_T$ and $u_N$ are Lagrange multipliers
Case $\mu = 0$

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Known solution techniques ($\mu \neq 0$)

1. reformulate as fixed point equation:

\[ P_L(r + u(r)) = r \]

and apply fixed-point operator successively [De Saxcé, Feng 91]

2. replace complementarity by nonlinear equation

\[ L \ni r \perp u \in L^* \iff \phi(u, r) = 0 \]

and apply Newton algorithm [Alart, Curnier 88]

Both solve (a-d) altogether iteratively
Our approach

To solve

\[
\begin{cases}
    Hr = Mv + f & (a) \\
    u = H^\top v + \mu Es & (b) \\
    L \ni r \perp u \in L^* & (c) \\
    s^i = \|u_T^i\| \forall i & (d)
\end{cases}
\]

- our approach:
  - keep (a-c) satisfied for fixed $s$ (inner problem)
  - solve (d) iteratively (outer problem)

- motivation: (a-c) is an “easy problem” (SOCP)
Inner problem

- Problem: solve with $s$ as a parameter

$\begin{cases} 
Hr = Mv + f \\
u = H^T v + \mu Es \\
L \ni r \perp u \in L^* 
\end{cases}$ (a) (b) (c)

- recognize KKT conditions of

$\begin{cases} 
\min J_s(r) := \frac{1}{2} r^T Wr - b_s^T r \\
r \in L 
\end{cases}$ (5)

with $W = H^T M^{-1} H$ and $b_s = H^T M^{-1} f - \mu Es$

- recognize also KKT conditions of

$\begin{cases} 
\min J(v) := \frac{1}{2} v^T M v + f^T v \\
H^T v + \mu Es \in L^* 
\end{cases}$ (6)
Inner problem

- Problem: solve with $s$ as a parameter

\[
\begin{cases}
Hr = Mv + f & (a) \\
u = H^T v + \mu Es & (b) \\
L \ni r \perp u \in L^* & (c)
\end{cases}
\]

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(6)
Non-smooth Newton

Idea: solve (d) with Newton algorithm

\[ s^i = \|u^i_T(s)\| \iff F(s) = s \quad (d) \]

and stabilize (damped Newton) by

\[ \min_{s \geq 0} \frac{1}{2} \|F(s) - s\|^2 \]

Issues:

- properties of \( u(s) \)
  - Lipschitz continuity
  - semi-smoothness
- properties of algorithm
  - ensure descent direction (stabilization)
  - local, global convergence
Non-smooth Newton

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Structure of algorithm

$u(s)$

SOCP
Inner problem

Newton
(+ line-search)
Outer problem

$s$
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3. **Existence, non-uniqueness**
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Existence

Fixed-point formulation

\[ s^i = \|u^i_T(s)\| \iff F(s) = s \quad (d) \]

- \( F(\cdot) \) bounded continuous on \( \mathbb{R}^n_+ \)
- apply Brouwer’s fixed-point theorem
- \( \Rightarrow \) existence of solutions
Non-uniqueness

Consider a bead stuck in a corner:

Two possible outcomes depending on initial conditions
Non uniqueness

Indeed, plotting \( s \rightarrow \frac{1}{2} \| F(s) - s \|^2 \) gives
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Convergence log (3D, 100 contacts)
Summary

- 3D friction problem as fixed-point
- exploit structure
- theoretical interest
- encouraging numerical results

Open questions and future work
- complete theoretical study
- compare with existing codes
- good implementation needed

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