Local analysis of dynamical systems and application to nonlinear waves

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Méthodes de dynamique non linéaire pour l'ingénierie des structures

Part II : center manifolds for maps

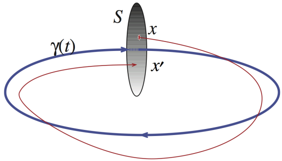
Outline :

- Introduction : main ideas, basic references
- Discrete spatial dynamics, unbounded infinite-dimensional maps
- Center manifold theorem for unbounded maps
- Application : time-periodic oscillations in FPU

Introduction

Problem : dynamics of an iterated map close to a fixed point.

Classical context : Poincaré map for an autonomous or periodic differential equation / PDE



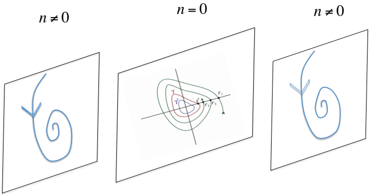
From : J.D. Meiss, Differential dynamical systems, SIAM '07

- Fixed point x_0 of the Poincaré map P $\Leftrightarrow$ periodic orbit γ of the flow.
- From local dynamics of P : stability of γ, local bifurcations.
- Can such informations be extracted from a lower-dim map ?

Introduction

Example : $N+1$ coupled oscillators $\quad \ddot{y}_n + f_n(y_n, \dot{y}_n) = \epsilon\, g_n(y, \dot{y})$

- Phase space in the uncoupled case $\epsilon = 0$:



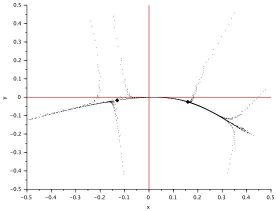
- A periodic orbit γ with oscillations localized near $n = 0$ persists for $\epsilon \ll 1$ under nondegeneracy conditions (Sepulchre and MacKay, Nonlinearity 10, '97).
- $\mathrm{Spec}(DP(x_0)) = 2N$ stable eigenvalues $(|\cdot| < 1) \cup \{\, \sigma_0 \,\}$
- If $\sigma_0 \approx 1$ (while stable spectrum remains far away) : local reduction to 1D map on a center manifold

Introduction

Example of a 1D center manifold for a 2D map :

$$
\begin{aligned}
x_{n+1} &= \mu - e^{-x_n} - \frac{1}{2}\,x_n\,y_n \\
y_{n+1} &= \frac{1}{2}\,(y_n - x_n^2)
\end{aligned}
$$

For $\mu = 1.01$, orbits close to the origin are attracted by a center manifold which contains a pair of (stable and unstable) fixed points :



Introduction

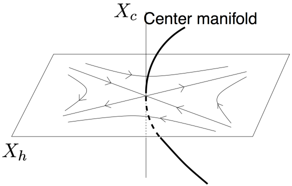
Local center manifolds for C^k maps ($k \geqslant 2$) :

$$u_{n+1} = F(u_n, \mu), \qquad F : \mathbb{R}^N \times \mathbb{R}^p \to \mathbb{R}^N \text{ is } C^k, \qquad F(0,0) = 0$$

$$\mathbb{R}^N = X_c \oplus X_h \text{ invariant under } L = D_u F(0,0)$$

Eigenvalues σ_k : for $L_{|X_c}$: $|\sigma_k| = 1$, for $L_{|X_h}$: $|\sigma_k| \neq 1$

Local dynamics : $u_{n+1} = L\, u_n$, $\mathbf{u_{n+1}} = \mathbf{F}(\mathbf{u_n}, \mu)$ $(\mu \approx 0)$



Introduction

Properties of the C^k center manifold $\mathcal{M}_\mu$ for $\mu \approx 0$:

- $\mathcal{M}_\mu$ locally invariant by $F(.,\mu)$
- $\mathcal{M}_\mu$ has same dimension as X_c, is tangent to X_c at $u = 0$ for $\mu = 0$
- $\mathcal{M}_\mu$ contains all orbits staying in some neighborhood of $u = 0$ for all $n \in \mathbb{Z}$
- If $|\sigma_k| < 1$ on X_h (i.e. no unstable eigenvalue in the hyperbolic part of the spectrum for $\mu = 0$) :
  $\mathcal{M}_\mu$ is locally exponentially attracting, and the stability of fixed points of $F(.,\mu)_{|\mathcal{M}_\mu}$ close to $u = 0$ is the same as for $F(.,\mu)$.

Introduction

Bibliography :

J. Carr, *Applications of center manifold theory*, Springer, 1981.

Case of infinite-dimensional C^k maps (in Banach spaces) :

- G. Iooss, *Bifurcation of maps and applications*, Math. Studies 36 (1979), Elsevier-North-Holland, Amsterdam.
- J. Marsden and M. McCracken, *The Hopf bifurcation and its applications*, Springer Verlag, NY, 1976.

Discrete spatial dynamics

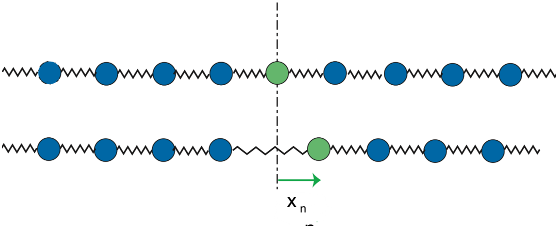
Provides an application of center manifold reduction involving unbounded infinite-dimensional maps

Fermi-Pasta-Ulam (FPU) model :

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \;\; n \in \mathbb{Z}$$

$$x_n(t) \in \mathbb{R}$$

Anharmonic interaction potential V : $V'(0) = 0, \; V''(0) > 0.$



Discrete spatial dynamics

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}$$

Invariances :

$x_n(t) \to x_n(t) + c \ \ (c \in \mathbb{R}), \quad x_n(t) \to -x_{-n}(t)$

- We want to determine time-periodic solutions (period T) close to $x_n = 0$.
- In particular breathers (spatially localized)

$$x_n(t+T) = x_n(t), \quad \lim_{n\to\pm\infty} \| x_n - c_\pm \|_{L^\infty} = 0, \ c_\pm \in \mathbb{R}$$

Discrete spatial dynamics

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}$$

$V'(0) = 0,\ V''(0) = 1$

- New variable : $y_n(\omega\, t) = V'(x_n(t) - x_{n-1}(t)), \quad T = 2\pi/\omega$
- Breather solutions satisfy $\lim_{n\to\pm\infty} \| y_n \|_{L^\infty} = 0$
- We search for y_n satisfying :

$$\int_0^{2\pi} y_n(t)\, dt = 0, \quad y_n(t + 2\pi) = y_n(t)$$

Reformulation of FPU :
$W = (V')^{-1}$, frequency ω = bifurcation parameter

$$\omega^2 \frac{d^2}{dt^2} W(y_n) = y_{n+1} - 2y_n + y_{n-1}, \quad n \in \mathbb{Z}$$

Discrete spatial dynamics

Notations : $H^0 = L^2_{per}(0, 2\pi)$ (square-integrable periodic functions)
Sobolev space $H^p_{per}(0, 2\pi)$: pth first derivatives in $L^2_{per}(0, 2\pi)$

$$H^p = \{\, y \in H^p_{per}(0, 2\pi) \,/\, y \text{ is even}, \int_0^{2\pi} y\, dt = 0 \,\}$$

Mapping for $Y_n = (y_{n-1}, y_n) \in D$, loop space $D = H^2 \times H^2$

$$\forall n \in \mathbb{Z}, \quad Y_{n+1} = F_\omega(Y_n) \quad \text{in} \quad X = H^2 \times H^0$$

$$F_\omega(y_{n-1}, y_n) = \Big(y_n \,,\, \omega^2 \frac{d^2}{dt^2} W(y_n) + 2y_n - y_{n-1} \Big)$$

$F_\omega : D \to X$ is C^k near $Y = 0$

- F_ω and T commute, $(T\,Y)(t) = Y(t + \pi)$
- Reversibility : Y_n solution $\Rightarrow R\,Y_{-n}$ solution, $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

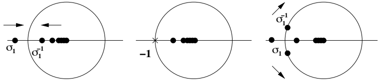
Discrete spatial dynamics

Linearized operator at $Y = 0$: $D \subset X \to X$ closed, unbounded

$$DF_\omega(0)(x,y) = \Big(y \, , \, (\omega^2 \frac{d^2}{dt^2} + 2)y - x \Big)$$

Eigenvalues σ_k, σ_k^{-1} $(k \geqslant 1)$: $\sigma^2 + (\omega^2 k^2 - 2)\sigma + 1 = 0$

Eigenvalues near the unit circle for $\omega \approx 2$:



$\omega > 2$ $\omega = 2$ $\omega < 2$

For $\omega = 2$: spectrum on the unit circle = double non semi-simple eigenvalue -1

$X = X_c \oplus X_h$

$\downarrow$

gen. eigenspace for $\sigma = -1$

$$\begin{cases} X_c = \text{Span } \{(\cos t, 0), (0, \cos t)\} \\ X_h = \text{Span } \{(\cos(kt), 0), (0, \cos(kt)), k \geqslant 2\} \end{cases}$$

Center manifolds for unbounded maps

$$\left.\begin{aligned} \forall n \in \mathbb{Z}, \quad u_n &\in D, \\ u_{n+1} = L\, u_n + N(u_n, \mu) \quad &\in X \end{aligned}\right\} \text{ Hilbert spaces}$$

$L : D \subset X \to X$ closed unbounded linear operator

Nonlinear term : $N : D \times \mathbb{R}^p \to X$ is C^k $(k \geqslant 2)$
$N(0,0) = 0, \quad D_u N(0,0) = 0.$

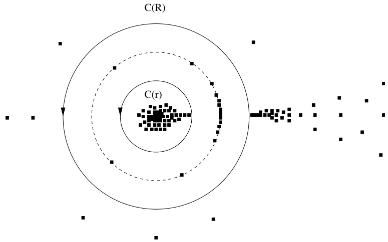
Parameter $\mu \in \mathbb{R}^p$, $\mu \approx 0$.

FPU : $Y_{n+1} = F_\omega(Y_n)$ $\left\{\begin{array}{l} L = DF_{\omega=2}(0), \ \mu = \omega^2 - 4 \\ N = F_\omega - L = O(\, \|Y_n\|_D^2 + \|Y_n\|_D\, |\mu|\,) \end{array}\right.$

Center manifolds for unbounded maps

SPECTRUM OF L :

$$\sigma(L) = \sigma_s \cup \sigma_c \cup \sigma_u,$$



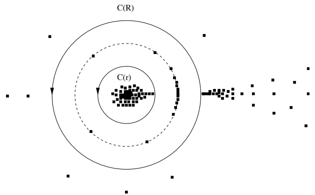
SPECTRAL SEPARATION :

$$\operatorname*{Sup}_{z \in \sigma_s} |z| < 1, \quad |z| = 1 \ \ \forall z \in \sigma_c, \quad \operatorname*{Inf}_{z \in \sigma_u} |z| > 1$$

Center manifolds for unbounded maps

SPECTRUM OF L :

$$\sigma(L) = \sigma_s \cup \sigma_c \cup \sigma_u,$$



spectral projections on stable / central subspace (regularizing) :

$$\begin{cases} \pi_s = \frac{1}{2i\pi}\int_{\mathcal{C}(r)} (zI - L)^{-1}\, dz, \\ \pi_c = \frac{1}{2i\pi}\int_{\mathcal{C}(R)} (zI - L)^{-1}\, dz - \pi_s \end{cases}$$

$X_c = \pi_c X \subset D$, $X_s = \pi_s X \subset D$,

$\pi_h = I_X - \pi_c$, $X_h = \pi_h X$, $D_h = \pi_h D$

Center manifolds for unbounded maps

$$u_{n+1} = L\, u_n + N(u_n, \mu) \qquad (E)$$

THEOREM 1 : Assume spectral separation for L

Then there exist neighborhoods of 0 : $\Omega \subset D$, $\Lambda \subset \mathbb{R}^p$,
a C^k local center manifold $\mathcal{M}_\mu \subset D$ $(\mu \in \Lambda)$:

- $\mathcal{M}_\mu$ same dimension as X_c, tangent to X_c at $u = 0$ for $\mu = 0$,

  $\mathcal{M}_\mu = \{\, y \in D \,/\, y = x + \psi(x,\mu),\ x \in X_c \cap \Omega \,\}$, $\psi : X_c \times \mathbb{R}^p \to D_h$

- $\mathcal{M}_\mu$ is locally invariant under $L + N(., \mu)$,

- (E) invariant under a linear isometry $\Rightarrow \mathcal{M}_\mu$ invariant under this isometry,

- (E) reversible mapping (+ technical assumptions) $\Rightarrow \mathcal{M}_\mu$ invariant under the reversibility symmetry.

Center manifolds for unbounded maps

$$u_{n+1} = L\, u_n + N(u_n, \mu) \qquad (E)$$

THEOREM 1 : (sequel) Assume spectral separation for L

$$\mathcal{M}_\mu = \{\, y \in D \,/\, y = x + \psi(x,\mu),\ x \in X_c \cap \Omega \,\}, \quad \psi : X_c \times \mathbb{R}^p \to D_h$$

- Local reduction of (E) :

$$\left.\begin{array}{l} (u_n) \text{ solution of } (E) \\ \\ u_n \in \Omega \text{ for all } n \in \mathbb{Z} \end{array}\right\} \Rightarrow u_n \in \mathcal{M}_\mu \text{ for all } n \in \mathbb{Z}$$

If dim $X_c < \infty$:
local infinite-dimensional problem $\Longleftrightarrow$ finite-dimensional mapping on $\mathcal{M}_\mu$

Center manifolds for unbounded maps

Reduced mapping on the center manifold : if $u_n \in \mathcal{M}_\mu$ for all $n \in \mathbb{Z}$ then $u_n^c = \pi_c u_n$ satisfies the C^k recurrence relation in X_c :

$$\forall n \in \mathbb{Z}, \quad u_{n+1}^c = f(u_n^c, \mu)$$

$$f(., \mu) = \pi_c (L + N(., \mu)) \circ (I + \psi(., \mu))$$

Functional equation satisfied by the reduction function ψ :

$$\psi(L_c x + \pi_c N(x + \psi(x, \mu), \mu), \mu) = L_h \psi(x, \mu) + \pi_h N(x + \psi(x, \mu), \mu)$$

To compute the Taylor expansion of ψ at $(x, \mu) = (0, 0)$:

- expand each side of the functional equation with respect to (x, μ) and identify terms of equal order
- $\Longrightarrow$ hierarchy of linear problems for the Taylor coefficients of ψ which can be solved by induction, starting from lowest order

General ideas of the proof

Cut-off on nonlinear terms : $N_\varepsilon(u,\mu) = N(u,\mu)\,\chi(\varepsilon^{-1}\|u\|_D)$

$$\chi : \mathbb{R} \to \mathbb{R} \text{ is } C^\infty, \quad \begin{cases} \chi(x) = 1 \text{ for } x \in [0,1], \\ \chi(x) = 0 \text{ for } x \geqslant 2. \end{cases}$$

Locally equivalent problem :

$$u_{n+1} = L\,u_n + N_\varepsilon(u_n,\mu) \quad \forall n \in \mathbb{Z}$$

Splitting on central and hyperbolic subspaces :

$$\begin{aligned} u^c_{n+1} &= L_c\,u^c_n + \pi_c\,N_\varepsilon(u_n,\mu), \qquad & u^c_n = \pi_c\,u_n,\ L_c = L_{|X_c} \\ u^h_{n+1} &= L_h\,u^h_n + \pi_h\,N_\varepsilon(u_n,\mu), \qquad & u^h_n = \pi_h\,u_n,\ L_h = L_{|X_h} \end{aligned}$$

Step 1 : corresponding affine equations $\quad f = (f_n)_{n\in\mathbb{Z}},\ u = (u_n)_{n\in\mathbb{Z}}$

Center manifolds for unbounded maps

Affine equation on X_c : $\quad u_{n+1}^c = L_c\, u_n^c + f_n^c, \qquad \forall n \in \mathbb{Z}$

$L_c,\ L_c^{-1} \in \mathcal{L}(X_c) \Rightarrow$ initial value problem has unique solution :

$$u_n^c = L_c^n\, u_0^c + (\, K_c\, f^c\,)_n, \qquad (K_c\, f)_n = \begin{cases} \sum_{k=0}^{n-1} L_c^{n-1-k}\, f_k & \text{for } n \geqslant 1, \\ 0 & \text{for } n = 0, \\ -\sum_{k=n}^{-1} L_c^{n-1-k}\, f_k & \text{for } n \leqslant -1. \end{cases}$$

Possible divergence of u : $f \in \ell_\infty(X_c) \not\Rightarrow u \in \ell_\infty(X_c)$. Appropriate spaces :

$f \in B_\nu(X_c) \Rightarrow u \in B_\nu(X_c)$ $\quad$ since $\lim_{k\to+\infty} \|L_c^{\pm k}\|_{\mathcal{L}(X_c)}^{1/k} = 1$

$$\nu \in (0,1), \quad B_\nu(X_c) = \{\ u \,/\, u_n \in X_c,\ \operatorname{Sup}_{n\in\mathbb{Z}}\ \nu^{|n|}\, \|u_n\|_{X_c} < +\infty\ \}$$

Center manifolds for unbounded maps

Affine equation on X_h : for any $f^h \in \ell_\infty(X_h)$, we solve

$$u^h \in \ell_\infty(D_h), \qquad u^h_{n+1} = L_h\, u^h_n + f^h_n \qquad \forall n \in \mathbb{Z}.$$

$L_h \,:\, D_h \subset X_h \to X_h$ unbounded. Unique bounded sol. $u^h = K_h\, f^h$

a) Existence :

$$u^h_n = \sum_{k=-\infty}^{+\infty} G_{n-k}\, f^h_k, \qquad G_q = \begin{cases} L_s^{q-1}\pi_s & \text{for } q \geqslant 1, \\ -(L_u^{-1})^{1-q}\pi_u & \text{for } q \leqslant 0. \end{cases}$$

Notations :

$$\sigma(L_h) = \sigma_s \cup \sigma_u = \sigma(L_s) \cup \sigma(L_u), \quad X_h = D_s \oplus X_u, \quad I_{X_h} = \pi_s + \pi_u.$$

$$D_s \subset D_h, \quad L_s = L_{|D_s} \in \mathcal{L}(D_s)$$

$$L_u \,:\, D_u \subset X_u \to X_u \text{ unbounded}, \qquad L_u^{-1} \in \mathcal{L}(X_u, D_u)$$

Spectral gap $\Rightarrow G_q \,:\, X_h \to D_h$, $\|G_q\|_{\mathcal{L}(X_h,D_h)} \leqslant \kappa\, r^{|q|}$, $r \in (0,1)$

Center manifolds for unbounded maps

b) Uniqueness : spectral separation $\Rightarrow$ for $f^h = 0$, nontrivial solutions $u^h \neq 0$ diverge exponentially as $n \to +\infty$ or $-\infty$.

Step 2 : non-local equation

$$u = L_c^n\, u_0^c + (K_c \pi_c + K_h \pi_h)\, N_\varepsilon(u, \mu)$$

Solved for $\varepsilon \approx 0$ and any fixed $(u_0^c, \mu) \in X_c \times \mathbb{R}^p$, with $\|\mu\| \leqslant \varepsilon^2$.
Contraction mapping theorem in $B_\nu(D) \Rightarrow$ unique solution

$$u_n = \phi_n^\varepsilon(u_0^c, \mu)$$

By uniqueness $u_{n+p} = \phi_{n+p}^\varepsilon(u_0^c, \mu) = \phi_n^\varepsilon(u_p^c, \mu)$. Fixing $n = 0$:

$$u_p = \phi_0^\varepsilon(u_p^c, \mu) \quad \forall p \in \mathbb{Z}.$$

$\phi_0^\varepsilon(., \mu)\,:\, X_c \to D$ continuous (C^k for $\varepsilon < \varepsilon_0(k)$, more technical).

Breathers in FPU

$$\forall n \in \mathbb{Z}, \qquad y_n \in H^2, \qquad \omega^2 \frac{d^2}{dt^2} W(y_n) = y_{n+1} - 2y_n + y_{n-1} \qquad (E)$$

$$H^2 = \{\, y \in H^2_{per}(0, 2\pi) \,/\, y \text{ is even}, \int_0^{2\pi} y\, dt = 0 \,\}$$

THEOREM 2 : Reduction near $y_n = 0$ and $\omega = 2$ (bif. at $\sigma = -1$)

If $y = (y_n)$ solution of (E), $\|y\|_{\ell_\infty(H^2)} + |\omega - 2|$ small enough,

then $y_n = \beta_n \cos t + \varphi_\omega(\beta_{n-1}, \beta_n)$, $\varphi_\omega : \mathbb{R}^2 \to H^2$ is C^k

$$\varphi_\omega = -\tfrac{1}{16}\, V^{(3)}(0) \cos(2t) \left(\beta_{n-1}\, \beta_n + \tfrac{1}{2}\, \beta_{n-1}^2 - \tfrac{7}{2}\, \beta_n^2 \right) + \text{ h.o.t.}$$

"Reduced" recurrence relation : invariances $n \to -n$, $\beta_n \to -\beta_n$

$$\beta_{n+1} + 2\beta_n + \beta_{n-1} = -4(\omega - 2)\, \beta_n + b\, \beta_n^3 + \text{ h.o.t.},$$

$$b = \frac{1}{2} V^{(4)}(0) - (V^{(3)}(0))^2$$

Breathers in FPU

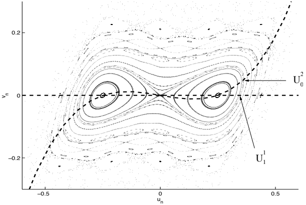
$\Rightarrow$ study of a reversible mapping in $\mathbb{R}^2$:

$$u_n = (-1)^n \beta_n, \qquad u_{n+1} - 2u_n + u_{n-1} = 4(\omega - 2)\, u_n - b\, u_n^3 + \text{ h.o.t.}$$

$$U_{n+1} = G_\omega(U_n),$$

$$U_n = (u_n, v_n), \qquad v_n = u_n - u_{n-1}.$$

Orbits of the map
for $b > 0$, $\omega > 2$, $\omega \approx 2$:



Continuum limit : $\mu = 4(\omega - 2) \approx 0$

$$u_n = \sqrt{\frac{\mu}{b}}\; u(n\sqrt{\mu}) + O(|\mu|) \quad \Rightarrow \quad u'' = u - u^3$$

Under this approx : $b > 0 \Rightarrow$ homoclinic orbits to 0 $\Rightarrow$ "breathers"

Breathers in FPU

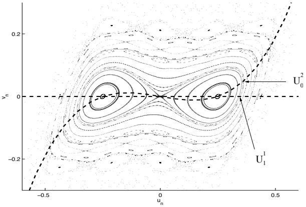
$\Rightarrow$ study of a reversible mapping in $\mathbb{R}^2$:

$$u_n = (-1)^n \beta_n, \qquad u_{n+1} - 2u_n + u_{n-1} = 4(\omega - 2)\, u_n - b\, u_n^3 + \text{ h.o.t.}$$

$$U_{n+1} = G_\omega(U_n),$$

$$U_n = (u_n, v_n), \qquad v_n = u_n - u_{n-1}.$$

Orbits of the map
for $b > 0, \quad \omega > 2, \quad \omega \approx 2$:



$(G_\omega R_i)^2 = I$, symmetries $R_1\,(u,v) = (u - v, -v)$, $R_2 = R_1\, G_\omega$
Dashed curves : fixed points of R_1 (axis $v = 0$) and R_2.

Reversible orbits homoclinic to 0 : $R_1 U^1_{-n+2} = U^1_n$, $R_2 U^2_{-n} = U^2_n$
$\Rightarrow$ "breathers"

Breathers in FPU

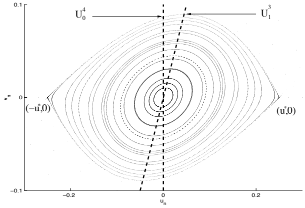
Reversible mapping in $\mathbb{R}^2$:

$$u_n = (-1)^n \beta_n, \qquad u_{n+1} - 2u_n + u_{n-1} = 4(\omega - 2)\, u_n - b\, u_n^3 + \text{ h.o.t.}$$

$$U_{n+1} = G_\omega(U_n),$$

$$U_n = (u_n, v_n), \qquad v_n = u_n - u_{n-1}.$$

Orbits of the map
for $b < 0, \quad \omega < 2, \quad \omega \approx 2$:



$(G_\omega R_i)^2 = I$, symmetries $R_3 = -R_1$, $R_4 = -R_2$
Dashed lines : fixed points of R_3 ($v = 2\,u$) and R_4 ($u = 0$).

$b < 0 \Rightarrow$ heteroclinic orbits : $R_3 U^3_{-n+2} = U^3_n, \quad R_4 U^4_{-n} = U^4_n$
$\Rightarrow$ "dark breathers"

Breathers in FPU

$$\frac{d^2x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \qquad y_n(t) = V'(x_n - x_{n-1})(t/\omega)$$

$$y_n \in H^2_{per}(0, 2\pi), \qquad \omega^2 \frac{d^2}{dt^2} W(y_n) = y_{n+1} - 2y_n + y_{n-1}, \quad n \in \mathbb{Z}$$

THEOREM 3 : $\exists$ solutions x_n, frequency $\omega \approx 2$, amplitude $O(|\omega - 2|^{1/2})$, "breathers" $(\omega > 2)$ or "dark breathers" $(\omega < 2)$.

a) If $\frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 > 0$: breathers y_n^1, y_n^2,

$$\lim_{n \to \pm\infty} \|y_n^i\|_{H^2} = 0, \quad y^1_{-n+1}(t) = y^1_n(t + \pi), \quad y^2_{-n}(t) = y^2_n(t)$$

b) If $\frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 < 0$: dark breathers y_n^3, y_n^4, homoclinic to a binary oscillation $y_n^0 = y(t + n\pi)$

$$\lim_{n \to -\infty} \|y_n^i - y_{n+1}^0\|_{H^2} = 0, \quad \lim_{n \to +\infty} \|y_n^i - y_n^0\|_{H^2} = 0$$

$$y^3_{-n+1} = y^3_n, \quad y^4_{-n}(t) = y^4_n(t + \pi)$$

Breathers in FPU

Principal part of y_n = slow spatial modulation of a standing wave of the linearized problem : $y_n(t) = (-1)^n \cos t$

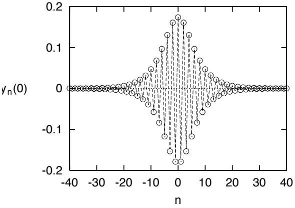


FIGURE: Breather

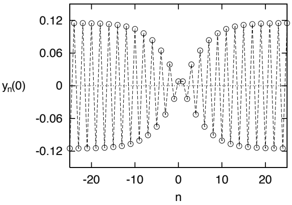


FIGURE: Dark breather

↑ Profiles for $\omega \approx 2$: (Sanchez-Rey, G. J., Cuevas, Archilla, '04)

- numerically computed solutions for polynomial potentials (circles)
- analytical approximations obtained using the reduced map (dashed line)

Breathers in FPU

$$\frac{d^2x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1})$$

Relative displacements and interaction forces :

$$z_n = x_n - x_{n-1}, \qquad y_n(\omega\, t) = V'(z_n)(t), \qquad \mu = \omega^2 - 4 \ll 1$$

Exact breather solutions : $z_n = (-1)^n u_n \cos(\omega t) + O(|\mu|)$

$$u_{n+1} + u_{n-1} - 2u_n = \mu\, u_n - b\, u_n^3 + \text{h.o.t.}, \quad (\mu, b > 0)$$

Principal part as $\mu \to 0$:

$$u_n = \sqrt{\frac{\mu}{b}}\; u(n\sqrt{\mu}) + O(|\mu|) \quad \text{with} \quad u'' = u - u^3$$

$$z_n(t) = (-1)^n \sqrt{\frac{2\mu}{b}} \frac{\cos \omega t}{\cosh(n\sqrt{\mu})} + O(|\mu|)$$ close to NLS approx.

References

- General center manifold theorem, application to FPU :

  G. J., Centre manifold reduction for quasilinear discrete systems, J. Nonlinear Sci. 13 (2003), 27-63.

- Oscillator chains including small inhomogeneities, local potentials :

  G. J., B. Sànchez-Rey and J. Cuevas, Reviews in Mathematical Physics 21 (2009), 1-59.

- Reduction near a spatially localized equilibrium :

  M. Georgi, Int. J. Dyn. Syst. Diff. Equations 2 (2009), 66-95.

- Diatomic FPU chains :

  G. J. and P. Noble, Physica D 196 (2004), 124-171.

  G. J. and M. Kastner, Nonlinearity 20 (2007), 631-657.

Part III : center manifolds for differential equations

Outline :

- Finite-dimensional case
- Differential equations in Banach spaces

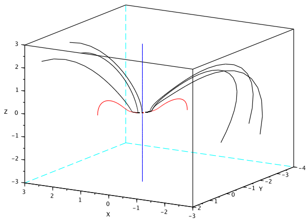
  (PDE, lattices, differential equations with delay and/or advance terms,...)
- Application : pulsating traveling waves in the Fermi-Pasta-Ulam model

Center manifolds for finite-dimensional ODE

Example (Lorenz system) :

$$\begin{aligned} x' &= y - x \\ y' &= x - y - x\,z \\ z' &= x\,y - z \end{aligned}$$

- spectrum of the linearization at 0 : $\{0\} \cup \{-1, -2\}$
- kernel spanned by $(1, 1, 0)^T$
- in a neighborhood of 0, trajectories attracted (exponentially) by a 1D center manifold (in red), 0 asymptotically stable :



Center manifolds for finite-dimensional ODE

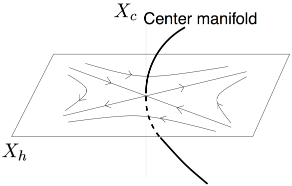
Local center manifolds for C^k $(k \geqslant 2)$ differential equations in $\mathbb{R}^n$:

$$(E) \quad u' = F(u,\mu), \qquad F : \mathbb{R}^N \times \mathbb{R}^p \to \mathbb{R}^N \text{ is } C^k, \qquad F(0,0) = 0$$

$$\mathbb{R}^N = X_c \oplus X_h \text{ invariant under } L = D_u F(0,0)$$

Eigenvalues σ_k : $\mathrm{Re}\,\sigma_k = 0$ for $L_c = L_{|X_c}$, $\mathrm{Re}\,\sigma_k \neq 0$ for $L_h = L_{|X_h}$

Local dynamics : $\quad u' = L\,u, \quad \mathbf{u}' = \mathbf{F}(\mathbf{u},\mu) \quad (\mu \approx 0)$



Center manifolds for finite-dimensional ODE

Properties of the C^k center manifold $\mathcal{M}_\mu$ for $\mu \approx 0$:

- $\mathcal{M}_\mu$ locally invariant by the flow
- $\mathcal{M}_\mu$ has same dimension as X_c, is tangent to X_c at $u = 0$ for $\mu = 0$
- $\mathcal{M}_\mu$ contains all orbits staying in some neighborhood of $u = 0$ for all $t \in \mathbb{R}$
- If $\mathrm{Re}\, \sigma_k < 0$ on X_h (i.e. no unstable eigenvalue for $\mu = 0$) : $\mathcal{M}_\mu$ is locally exponentially attracting, and the stability of equilibria close to $u = 0$ is determined by the flow on $\mathcal{M}_\mu$.
- (E) invariant under a linear isometry $\Rightarrow \exists\ \mathcal{M}_\mu$ invariant under this isometry,
- (E) reversible (i.e. $F(., \mu)$ anticommutes with a symmetry) $\Rightarrow \exists\ \mathcal{M}_\mu$ invariant under the reversibility symmetry.

Center manifolds for finite-dimensional ODE

Notations :

- $F = L + N, \quad N(u, \mu) = O(|\mu| + \|u\|^2)$
- π_c, π_h : spectral projections on X_c, X_h
- $\mathcal{M}_\mu$ locally the graph of $\psi(., \mu)$: $X_c \to X_h$

Reduced equation on the center manifold :
if $u(t) \in \mathcal{M}_\mu$ for all $t \in \mathbb{R}$ then $u_c = \pi_c u$ satisfies the reduced equation in X_c :

$$u_c' = f(u_c, \mu)$$

$$f(., \mu) = \pi_c \,(L + N(., \mu)) \circ (I + \psi(., \mu))$$

Center manifolds for finite-dimensional ODE

The reduction function ψ satisfies :

$$(P) \quad D_x\psi(x,\mu)\, f(x,\mu) = L_h\, \psi(x,\mu) + \pi_h\, N(x+\psi(x,\mu),\mu)$$

- If $\dim X_c \geqslant 2$ then (P) corresponds to a PDE.
- Interpretation of (P) : vector field $L + N(.,\mu)$ tangent to the center manifold

To compute the Taylor expansion of ψ at $(x,\mu) = (0,0)$:

- expand each side of (P) with respect to (x,μ) and identify terms of equal order
- $\Longrightarrow$ hierarchy of linear problems for the Taylor coefficients of ψ which can be solved by induction, starting from lowest order
- if the parameterization of $\mathcal{M}_\mu$ is changed by allowing ψ to have a component of X_c, the reduced equation may be greatly simplified (normal form).

Center manifolds for finite-dimensional ODE

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- J. Carr, *Applications of center manifold theory*, Springer, 1981.
- A. Vanderbauwhede, *Centre manifolds, normal forms and elementary bifurcations*, Dynamics Reported 2, (U. Kirchgraber and H.O. Walther, eds) John Wiley and Sons Ltd and B.G. Teubner (1989), p. 89-169.
- G. Iooss, M. Adelmeyer, *Topics in bifurcation theory and applications*, Adv. Ser. Nonlinear Dynamics 3, World Sci. (1992).

Center manifolds in infinite dimensions

General framework : differential equation in a Banach space X :

$$\frac{du}{dt} = L\,u + N(u,\mu) \quad \mu \in \mathbb{R}^p \text{ small parameter}$$

Assumptions :

- Consider three Banach spaces with continuous embeddings : $D \subset Y \subset X$
- Linear term $L \in \mathcal{L}(D, X)$
- Nonlinear term $N \in C^k(D \times \mathbb{R}^p, Y)$ $(k \geqslant 2)$, $N(0,0) = 0$, $D_u N(0,0) = 0$
- $u(t) \in D$, $\frac{du}{dt}(t) \in X$

Applications :

PDE, lattices, differential equations with delay, advance-delay

Center manifolds in infinite dimensions

Example :

$$u_t = u_{xx} + u + \mu\, u - u\,(u_x)^2, \quad x \in (0, \pi)$$

with boundary conditions $u = 0$ at $x = 0$ and $x = \pi$

- Identification $u(x,t) \to [u(t)](x)$.
- Basic space : $X = L^2(0,\pi)$. Domain : Sobolev space $D = H^2(0,\pi) \cap H_0^1(0,\pi)$.
- We search for $u \in C^0(\mathbb{R}, D) \cap C^1(\mathbb{R}, X)$ solution of

$$\frac{du}{dt} = L\,u + N(u,\mu)$$

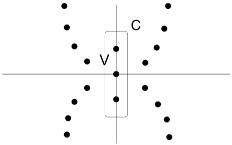
with $L = \frac{d^2}{dx^2} + 1$,
$N(u,\mu) = \mu\, u - u\,(u_x)^2 \,:\, D \times \mathbb{R} \to H_0^1(0,\pi) = Y$

Center manifolds in infinite dimensions

Assumption 1 : spectral separation $\sigma(L) = \sigma_c \cup \sigma_h$

$$\sigma_c \subset i\,\mathbb{R}, \quad \inf_{\lambda \in \sigma_h} |\mathrm{Re}\,\lambda| > 0$$

Assumption 2 : σ_c consists of a finite number of eigenvalues with finite multiplicities. $\quad (X_c := \bigoplus$ generalized eigenspaces $\subset D)$



Spectral projection on X_c : $\pi_c = \frac{1}{2i\pi} \int_{\mathcal{C}} (zI - L)^{-1}\, dz$

Notations : $\pi_h = I - \pi_c,\ X_h = \pi_h X,\ D_h = \pi_h D,\ Y_h = \pi_h Y$

Center manifolds in infinite dimensions

Assumption 3 : on the affine equation on X_h

$$\frac{du_h}{dt} = L\, u_h + f_h(t)$$

For all $f_h \in C^0_{\mathrm{bounded}}(\mathbb{R}, Y_h)$, there exists a unique solution $u_h \in C^0_{\mathrm{bounded}}(\mathbb{R}, D_h)$ and the map $f_h \mapsto u_h$ is continuous.

- Automatic if $L \in \mathcal{L}(X)$ with spectral separation, in particular in finite dimension :

$$u_h = \int_{\mathbb{R}} G(t-s)\, f_h(s)\, ds, \quad G(\tau) = \begin{cases} e^{L_s \tau}\pi_s \text{ for } \tau > 0 \\ -e^{L_u \tau}\pi_u \text{ for } \tau < 0 \end{cases}$$

- Tools : semigroup theory (resolvent estimates), transform techniques (Fourier, Laplace)

Center manifolds in infinite dimensions

Under assumptions $1, 2, 3$ on the linear problem, there exists for $\mu \approx 0$ a C^k local center manifold $\mathcal{M}_\mu$ (same dimension as X_c, tangent to X_c at $u = 0$ for $\mu = 0$) satisfying :

- $\mathcal{M}_\mu$ locally invariant by the flow (well-defined on the finite-dimensional center manifold)
- $\mathcal{M}_\mu$ contains all orbits staying in some neighborhood of $u = 0$ in D for all $t \in \mathbb{R}$
- $\mathcal{M}_\mu$ invariant under the symmetries of the evolution problem (isometries in D)
- If $\mathrm{Re}\,\sigma(L) < 0$ on X_h (i.e. no unstable eigenvalue for $\mu = 0$), and if the homogeneous linear initial value problem on D_h is well posed for $t \geqslant 0$, with $u = 0$ exponentially asymptotically stable in D_h, then $\mathcal{M}_\mu$ is locally exponentially attracting.

Center manifolds in infinite dimensions

Example (continued) :

$$u_t = u_{xx} + u + \mu\, u - u\,(u_x)^2, \quad x \in (0,\pi)$$

with boundary conditions $u = 0$ at $x = 0$ and $x = \pi$

$L = \frac{d^2}{dx^2} + 1$ with the above boundary conditions

$\sigma(L)$: simple eigenvalues $1 - k^2$ $(k \geqslant 1)$, eigenvectors $\sin(k\,x)$

$\sigma_c = \{0\}$, π_c = orthogonal projection (wrt $(.,.)_{L^2}$) on $X_c = \mathbb{R} \sin x$

Solution to the affine equation :

$$[u_h(t)](x) = \sum_{k\geqslant 2} \sin(k\,x) \int_{-\infty}^{t} e^{(1-k^2)(t-s)} b_k(s)\, ds$$

for $[f_h(t)](x) = \sum_{k\geqslant 2} \sin(k\,x)\, b_k(t)$

Center manifolds in infinite dimensions

Example (continued) :

$$u_t = u_{xx} + u + \mu\, u - u\,(u_x)^2, \quad x \in (0, \pi)$$

with boundary conditions $u = 0$ at $x = 0$ and $x = \pi$

For $\mu \approx 0$, there exists a one-dimensional local center manifold

$$\mathcal{M}_\mu = \{\, u = A \sin x + \psi(A, \mu),\ A \in (-\rho, \rho)\,\}$$

$\psi : \mathbb{R}^2 \to (\sin x)^\perp \cap D$, $\psi(A, \mu) = O(|A|^3 + |A\,\mu|)$ is odd in A

Reduced equation :

$$A' = \mu\, A - \frac{1}{4} A^3 + \mathrm{h.o.t.}$$

$\Longrightarrow$ supercritical pitchfork bifurcation (invariance $A \to -A$).

Center manifolds in infinite dimensions

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- G. J., Y. Sire, *Center manifold theory in the context of infinite one-dimensional lattices*, in : The Fermi-Pasta-Ulam Problem. A Status Report, G. Gallavotti Ed., Lecture Notes in Physics 728 (2008), p. 207-238.

Application : pulsating traveling waves in FPU

$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \ n \in \mathbb{Z}$$

$(V'(0) = 0, V''(0) = 1)$

We look for pulsating traveling waves :

$$u_n(t) = u_{n-p}(t - p\,\tau), \quad \text{for fixed } p \geqslant 1 \text{ and } \tau > 0$$

Formulation in a frame moving at constant velocity :
$u_n(t) = y_n(x)$, $x = n - t/\tau$
Advance-delay differential equation for $y_n(x) = u_n(\tau(n-x))$:

$$\frac{1}{\tau^2}\frac{d^2 y_n}{dx^2} = V'(y_{n+1}(x+1) - y_n(x)) - V'(y_n(x) - y_{n-1}(x-1)).$$

$$y_{n+p}(x) = y_n(x)$$

Pulsating traveling waves in FPU

Formulation as an infinite-dimensional *reversible* evolution pb. :

Additional variables : $Y_n(x,v) = y_n(x+v), \quad \xi_n = \frac{dy_n}{dx}$

$$\begin{aligned}
\frac{dy_n}{dx} &= \xi_n, \\
\frac{d\xi_n}{dx} &= \tau^2 [\, V'(Y_{n+1}(x,1) - y_n(x)) - V'(y_n(x) - Y_{n-1}(x,-1)) \,], \\
\frac{\partial Y_n}{\partial x} &= \frac{\partial Y_n}{\partial v}
\end{aligned}$$

Evolution problem for $U(x) = (U_n(x))_{n\in\mathbb{Z}}$,

$$U_n = (y_n, \xi_n, Y_n(v))^T, \quad U_{n+p} = U_n, \quad Y_{n|v=0} = y_n$$

Pulsating traveling waves in FPU

$$\frac{dU}{dx} = L_\tau U + \tau^2 M(U)$$

$\mathbb{D} := D(L_\tau)$: sequences $U = (U_n)_{n\in\mathbb{Z}}$ in $\mathbb{R}^2 \times C^1([-1,1])$, with period p, general term $U_n = (y_n, \xi_n, Y_n(v))^T$ with $Y_{n|v=0} = y_n$.

$$(L_\tau U)_n = \begin{pmatrix} \xi_n \\ \tau^2(\delta_1 Y_{n+1} - 2y_n + \delta_{-1} Y_{n-1}) \\ \frac{dY_n}{dv} \end{pmatrix}$$

$M(U) : \mathbb{D} \to \mathbb{D}$,
$M(U) = O(\|U\|^2)$ as $U \to 0$.

Pulsating traveling waves in FPU

- Reversibility symmetry R

$$(\mathcal{R}\, U)_n = (-y_{-n}, \xi_{-n}, -Y_{-n}(-v))^T.$$

$U(x)$ is a solution $\Longrightarrow \mathcal{R}U(-x)$ is a solution

- Index shift σ

$$(\sigma U)_n = U_{n+1}$$

$U(x)$ is a solution $\Longrightarrow \sigma U(x)$ is a solution

- First integral (x plays the role of time !)

$$\mathcal{I}_\tau(U) = \frac{1}{p}\sum_{n=1}^{p}\Big(\xi_n - \tau^2\int_0^1 V'(Y_{n+1}(v) - Y_n(v-1))\, dv\Big)$$

Originates from the invariance $y_n \to y_n + c$

Pulsating traveling waves in FPU

Spectrum of L_τ : isolated eigenvalues, finite multiplicities

$$z \text{ eigenvalue } \Leftrightarrow \prod_{m=0}^{p-1} \Big[\frac{z^2}{\tau^2} + 2(1 - \cosh(z - 2i\pi m/p)) \Big] = 0$$

Spectrum on the imaginary axis given by : $z = i\lambda$ ($\lambda \in \mathbb{R}$),

$$\frac{|\lambda|}{\tau} = 2 \Big| \sin\Big(\frac{\lambda}{2} - \pi \frac{m}{p}\Big) \Big|, \quad m \in \{0, \ldots, p-1\}.$$

Particle displacements : normal modes

$$u_n(t) = y_n(n - t/\tau) = a\, e^{i\lambda(n - t/\tau)}\, e^{-i n (2\pi m/p)} + \text{c.c.} = a\, e^{i(qn - \omega t)} + \text{c.c.},$$

with $|\omega| = 2|\sin(q/2)|$, $\quad q = \lambda - 2\pi m/p$, $\quad \omega = \frac{\lambda}{\tau}$.

Pulsating traveling waves in FPU

Spectrum of L_τ near the imaginary axis

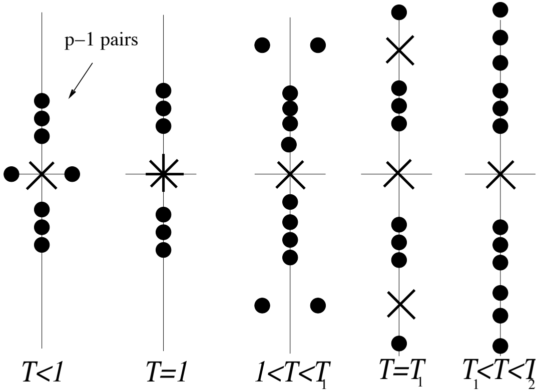


FIGURE: Eigenvalue : $\bullet$ = simple, $\times$ = double, $*$ = quadruple.

Pulsating traveling waves in FPU

The critical parameter values $1 < \tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$ (and bifurcating eigenvalues $i\lambda$) are given by :

$$\tau = \sqrt{1 + \frac{\lambda^2}{4}}$$

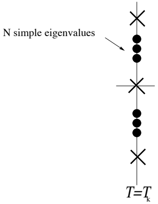
$$\frac{\lambda}{2} = \tan\left(\frac{\lambda}{2} - \pi \frac{m}{p}\right), \quad m \in \{0, \ldots, p-1\}.$$

For the corresponding normal modes :

$\frac{1}{\tau_k} = w'(q)$ (group velocity), $\omega'(q)(q + 2\pi \frac{m}{p}) = \omega(q)$.

Pulsating traveling waves in FPU

Spectrum of $L = L_{\tau_k}$ on the imaginary axis :



- $N = p + 2(k-1)$ pairs of simple eigenvalues $\pm i\lambda_1, \ldots, \pm i\lambda_N$,
  $\lambda_j \to m = m_j$, eigenvector ζ_j,
- 2 pairs of double eigenvalues $\pm i\lambda_0$,
  $\lambda_0 \to m = m_0$, eigenvector ζ_0, generalized eigenvector η_0,
- double eigenvalue 0,
  $\lambda = 0 \to m = 0$, eigenvector χ_0, generalized eigenvector χ_1.

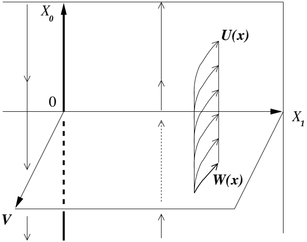
$\Rightarrow$ dimension of the central subspace $= 2N + 6$

Pulsating traveling waves in FPU

solution $y_n(x) = ax + b \;\Rightarrow\;$ solution $U(x) = (ax+b)\chi_0 + a\chi_1$
invariance $y_n \to y_n + q \;\Rightarrow\;$ invariance $U \to U + q\chi_0$

Invariant subspaces under L_{τ_k} :
$\mathbb{D} = \mathrm{Vect}(\chi_0, \chi_1) \oplus \mathbb{D}_1$

$$\begin{aligned} U(x) &= q(x)\,\chi_0 + \underbrace{d(x)\,\chi_1 + V(x)} \\ &= q(x)\,\chi_0 + \qquad W(x) \end{aligned}$$



$$\begin{aligned} \frac{dq}{dx} &= d \\ \frac{dW}{dx} &= \tilde{L}_\tau W + \tau^2 M(W) \end{aligned}$$

Pulsating traveling waves in FPU

Small amplitude solutions : $\underset{x \in \mathbb{R}}{\text{Sup}} \; \| W(x) \|_{\mathbb{D}} \approx 0$

THEOREM 1 :

For $\tau \approx \tau_k$, small amplitude $\subset$ $2N + 6$-dim center manifold :

$$\begin{aligned} U(x) &= A(x)\zeta_0 + B(x)\eta_0 + \sum_{j=1}^{N} C_j(x)\zeta_j + c.c. + D(x)\chi_1 + q(x)\chi_0 \\ &\quad + \psi(A(x), B(x), C(x), \bar{A}(x), \bar{B}(x), \bar{C}(x), D(x), \tau), \end{aligned}$$

with $C = (C_1, \dots, C_N)$.

Coordinates of solutions :

$$(A, B, C_1, \dots, C_N, \bar{A}, \bar{B}, \bar{C}_1, \dots, \bar{C}_N, D, q) \in \mathbb{C}^{2N+4} \times \mathbb{R}^2$$

$$\psi \in C^m(\mathbb{C}^{2N+4} \times \mathbb{R}^2, \mathbb{D}), \quad \psi(0, \tau) = 0, D\psi(0, \tau_k) = 0.$$

Pulsating traveling waves in FPU

THEOREM 2 : Normal form of order 3
The center manifold can be parameterized locally in order to have (for $\|W\|_{\mathbb{D}} \approx 0$, $\tau \approx \tau_k$)

$$\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D) + h.o.t.,\\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D) + h.o.t.,\\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D) + h.o.t., \quad (j = 1, \dots N)\\
\frac{dD}{dx} &= 0,\\
\frac{dq}{dx} &= D + \chi_1^*(\psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau)).
\end{aligned}$$

Reversibility symmetry $\mathcal{R}$: $(A, B, C, D, q) \mapsto (\bar{A}, -\bar{B}, \bar{C}, D, -q)$
Invariance under
$\sigma = \text{diag}(e^{-2i\pi\frac{m_0}{p}}, e^{-2i\pi\frac{m_0}{p}}, e^{-2i\pi\frac{m_1}{p}}, \dots, e^{-2i\pi\frac{m_N}{p}}, 1, 1)$

Pulsating traveling waves in FPU

THEOREM 2 : Normal form of order 3

The center manifold can be parameterized locally in order to have (for $\|W\|_{\mathbb{D}} \approx 0$, $\tau \approx \tau_k$)

$$
\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D) + h.o.t.,\\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D) + h.o.t.,\\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D) + h.o.t., \quad (j = 1, \dots N)\\
\frac{dD}{dx} &= 0,\\
\frac{dq}{dx} &= D + \chi_1^*(\psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau)).
\end{aligned}
$$

χ_1^* : linear form (coordinate along χ_1).

Pulsating traveling waves in FPU

THEOREM 2 : Normal form of order 3

The center manifold can be parameterized locally in order to have (for $\|W\|_{\mathbb{D}} \approx 0$, $\tau \approx \tau_k$)

$$
\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D) + h.o.t., \\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D) + h.o.t., \\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D) + h.o.t., \quad (j = 1, \ldots N) \\
\frac{dD}{dx} &= 0, \\
\frac{dq}{dx} &= D + \chi_1^*(\psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau)).
\end{aligned}
$$

Principal part : cubic polynomial in A, B, C, complex conjugates, and D.

Pulsating traveling waves in FPU

THEOREM 2 : Normal form of order 3

The center manifold can be parameterized locally in order to have (for $\|W\|_{\mathbb{D}} \approx 0$, $\tau \approx \tau_k$)

$$\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D) + h.o.t.,\\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D) + h.o.t.,\\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D) + h.o.t., \quad (j = 1, \dots N)\\
\frac{dD}{dx} &= 0,\\
\frac{dq}{dx} &= D + \chi_1^*(\psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau)).
\end{aligned}$$

Higher order terms are independent of q.

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Truncated normal form

$$\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D), \\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D), \\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j\mathcal{Q}_j(|A|^2, I, Q, D), \quad (j = 1, \dots N) \\
\frac{dD}{dx} &= 0.
\end{aligned}$$

First integrals : $D,\ Q = (|C_1|^2, \dots, |C_N|^2),\ I = i(A\bar{B} - \bar{A}B)$.

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We fix D, $Q = (|C_1|^2, \dots, |C_N|^2)$

$\rightarrow$ 1 :1 resonance with reversibility. Integrable system (Iooss-Pérouème '93).

$$\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D), \\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D).
\end{aligned}$$

For $\tau \approx \tau_k$ and $|D| + \|C\|^2 \ll |\tau - \tau_k|$:

$$\begin{aligned}
\mathcal{P} &= p_0(\tau) + r\,|A|^2 + f\,I + h.o.t. \\
\mathcal{S} &= s_0(\tau) + s\,|A|^2 + g\,I + h.o.t.
\end{aligned}$$

$$r, s, f, g, p_0(\tau), s_0(\tau) \in \mathbb{R}, \quad p_0(\tau_k) = s_0(\tau_k) = 0.$$

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$$\frac{dA}{dx} = i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D),$$
$$\frac{dB}{dx} = i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D).$$

For $\tau \approx \tau_k$ and $|D| + \|C\|^2 \ll |\tau - \tau_k|$:

$$\mathcal{P} = p_0(\tau) + r\,|A|^2 + f\,I + h.o.t.$$
$$\mathcal{S} = s_0(\tau) + s\,|A|^2 + g\,I + h.o.t.$$

$V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4 + \text{h.o.t}, \quad b = 3\beta - 4\alpha^2$
Wave velocity : $c_k = \frac{1}{\tau_k}$, $0 < c_k < 1$, c_k dense in $[0,1]$ for $p, k \geqslant 1$.

$$s = -16\,[b - c_k^2(b + 2\alpha^2)]$$

Case $s < 0 \Rightarrow$ localized solutions, agrees with NLS (Tsurui '72)

Pulsating traveling waves in FPU

Truncated system, $\tau \approx \tau_k$ with $\tau < \tau_k$, $D \approx 0$ with $|D| \ll |\tau - \tau_k|$

$$
\begin{aligned}
\frac{dA}{dx} &= i\lambda_0 A + B + iA\mathcal{P}(|A|^2, I, Q, D), \\
\frac{dB}{dx} &= i\lambda_0 B + [iB\mathcal{P} + A\mathcal{S}](|A|^2, I, Q, D), \\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j \mathcal{Q}_j(|A|^2, I, Q, D), \quad (j = 1, \ldots N) \\
\frac{dD}{dx} &= 0
\end{aligned}
$$

$s < 0 \;\Rightarrow \exists$ homoclinic orbits to N-dim tori, $N = p + 2(k-1)$.
Approximate solutions of the FPU system :

$$u_n(t) \approx A(n - t/\tau)\, e^{-2i\pi m_0 n/p} + \sum_{j=1}^{N} \left(C_j(n - t/\tau)\, e^{-2i\pi m_j n/p} \right) + c.c. + q(n - t/\tau)$$

with $\frac{dq}{dx} = D - 8\alpha\, |A|^2$.

Pulsating traveling waves in FPU

Homoclinic orbits to 0 :

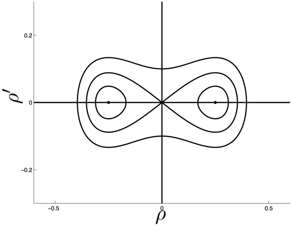
$$A = \rho(x)\, e^{i(\lambda_0 x + \psi(x))}, \quad B = \rho'(x)\, e^{i(\lambda_0 x + \psi(x))}$$

$$\rho'' = s_0(\tau)\rho + s\,\rho^3$$

$\longrightarrow$

Case $s < 0$

$(s_0(\tau < \tau_k) > 0)$



$$\psi' = p_0(\tau) + r\,\rho^2$$

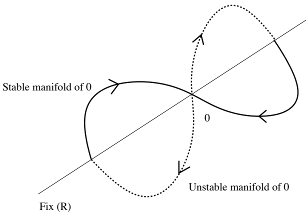
Pulsating traveling waves in FPU

Generic non-persistence of reversible homoclinics to 0 : heuristic

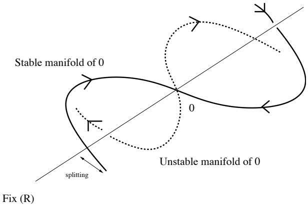
Case $D = 0$, normal form for $A, B, C_1, \ldots, C_N$.

Phase space : dimension $2N + 4$. Stable manifold of 0 : dim 2.

Reversibility symmetry R : $\dim \mathrm{Fix}(R) = N + 2$.



Truncated normal form



Complete normal form

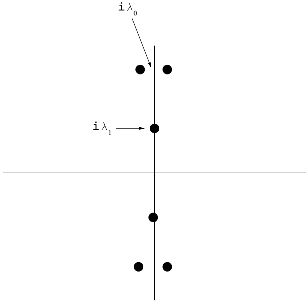
Stable manifold $\cap \mathrm{Fix}(R)$: $3N + 4$ conditions. $\Rightarrow$ codimension N $(N \geqslant p)$

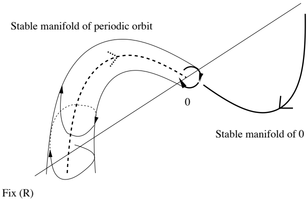
Pulsating traveling waves in FPU

Rigorous results : reversible $(i\lambda_0)^2(i\lambda_1)$ resonance ($N = 1$, $\tau \approx \tau_1$)

-case $p = 1$ (traveling waves)

-case $p = 2$, V is even, invariance under $-\sigma\ :\ (y_n) \mapsto -(y_{n+1})$





- Splitting distance of $W^s(0)$ and $\mathrm{Fix}(R)$ exponentially small in $|\tau - \tau_1|$ (and does not vanish generically) : Lombardi '00
- $\exists$ reversible solutions of the complete normal form homoclinic to periodic orbits with amplitudes $O(e^{-c/|\tau-\tau_1|^{1/2}})$ ($c > 0$).

Pulsating traveling waves in FPU

THEOREM 3 : Assume V is even and $V^{(4)}(0) > 0$
Exact FPU solutions : traveling breathers $u_n(t) = (-1)^n y(n - t/\tau)$ superposed at ∞ on periodic oscillations, with $\tau \approx \tau_1 \approx 3$ $(\tau < \tau_1)$

$$u_n(t) = \underbrace{(-1)^n A\,(n - t/\tau)}_{\text{Pulse}} + \underbrace{(-1)^n C_1\,(n - t/\tau)}_{\text{Periodic wave}} + c.c. + \text{h.o.t.}$$

*Pulse : modulation of a plane wave with wave number $q_0 \approx 2.5$,
*Periodic wave : modulated plane wave, wave number $q_1 \approx 0.8$

Families of reversible solutions : $\begin{cases} -u_{-n}(-t) = u_n(t) & \text{(reversible under } \mathcal{R}) \\ u_{-n}(-t) = u_n(t) & \text{(reversible under } -\mathcal{R}) \end{cases}$

For fixed τ (and up to a phase shift), each solution family is parameterized by the amplitude of the limiting periodic orbit, with lower bound $O(e^{-c/|\tau-\tau_1|^{1/2}})$ $(c > 0)$.

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