Epigraphical splitting for solving constrained convex formulations of inverse problems with proximal tools

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Grenoble Optimization Day
November 5th, 2014
Collaboration

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**Motivation**

\[ \mathbf{x} \in \mathbb{R}^{\bar{N}} \]
\[ A\mathbf{x} \in \mathbb{R}^{N} \]
\[ z = \mathcal{P}_\alpha(A\mathbf{x}) \]

- \( \mathbf{x} \): original image
- \( A \): linear operator from \( \mathbb{R}^{\bar{N}} \) to \( \mathbb{R}^{N} \)
- \( \mathcal{P}_\alpha \): effect of noise where \( \alpha > 0 \) is the scaling parameter
- \( z \): degraded image of size \( N \)
Motivation

$x \in \mathbb{R}^N$

$A\overline{x} \in \mathbb{R}^N$

$z = P_\alpha(A\overline{x})$

- $\overline{x}$: original image

**Assumption:** sparse after some appropriate transform

- $A$: linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$

- $P_\alpha$: effect of noise where $\alpha > 0$ is the scaling parameter

- $z$: degraded image of size $N$
Motivation

\[ \bar{x} \in \mathbb{R}^{\bar{N}} \quad \text{and} \quad A\bar{x} \in \mathbb{R}^{N} \quad \text{and} \quad z = \mathcal{P}_{\alpha}(A\bar{x}) \]

- \( \bar{x} \): original image
- **Assumption**: sparse after some appropriate transform
- \( A \): linear operator from \( \mathbb{R}^{\bar{N}} \) to \( \mathbb{R}^{N} \)
- \( \mathcal{P}_{\alpha} \): effect of noise where \( \alpha > 0 \) is the scaling parameter
- \( z \): degraded image of size \( N \)

**Objective**: recover \( \bar{x} \) from the observations \( z \)
Motivation

\[
\hat{x} \in \text{Argmin}_{x \in \mathbb{R}^N} g(Ax, z) \quad \text{Data fidelity term}
\]

\[
+ \lambda f(x) \quad \text{Regularization term}
\]

where $\lambda > 0$
Motivation: Existing works – Gaussian noise

Regularized approach

$$\min_{x \in \mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$$

[Tikhonov, 1963]

Constrained approach

$$\min \quad f(x)$$

$$\|Ax - z\|^2 \leq \eta$$

[Combettes, Trussell, 1991]
### Motivation: Existing works – Gaussian noise

<table>
<thead>
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| \[
\min_{x \in \mathbb{R}^N} \| Ax - z \|^2 + \lambda f(x) \]

[Tikhonov, 1963]  

→ Gradient-based methods  

| \[
\min f(x) \mid \| Ax - z \|^2 \leq \eta \]

[Combettes, Trussell, 1991]  

→ POCS [Trussell, Civanlar, 1984]  
→ Subgradient projections [Luo, Combettes, 1999]
### Motivation: Existing works – Gaussian noise

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<td>$\min_{x \in \mathbb{R}^N} |Ax - z|^2 + \lambda f(x)$</td>
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<td>[Tikhonov, 1963]</td>
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- $f = \|F \cdot \|^2$
  - → Gradient-based methods
  - → POCS [Trussell, Civanlar, 1984]
  - → Subgradient projections [Luo, Combettes, 1999]

- $f(x) = \sum_i |(Fx)^{(i)}|_1$
  (where $F$ is a wavelet transform, a frame)
  - → Proximal methods [Combettes, Pesquet, 2011]
  - → Proximal methods [Combettes, Pesquet, 2011]
Motivation: Existing works – Gaussian noise

Regularized approach

\[
\min_{x \in \mathbb{R}^N} \| Ax - z \|^2 + \lambda f(x)
\]

[Tikhonov, 1963]

Constrained approach

\[
\min_{f(x) \leq \eta} \| Ax - z \|^2
\]
Motivation: Existing works – Gaussian noise

Regularized approach

\[
\min_{x \in \mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)
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[Tikhonov, 1963]

Constrained approach

\[
\min_{f(x) \leq \eta} \|Ax - z\|^2
\]

If \( f = \| \cdot \|_1, p = \sum_{b \in \mathbb{L}} \| B_b \cdot \| \) with \( p \geq 1 \)

\( \rightarrow \) block sparsity measure:

for every \( b \in \mathbb{L} \subset \mathbb{K} \), \( B_b \) is a block selection transform.
## Motivation: Existing works – Gaussian noise

### Regularized approach

\[
\min_{x \in \mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)
\]

[Tikhonov, 1963]

\[
\text{If } f = \| \cdot \|_1, p = \sum_{b \in L} \| B_b \cdot \| \text{ with } p \geq 1 \rightarrow \text{block sparsity measure: }
\]

for every \( b \in L \subset K \), \( B_b \) is a block selection transform.

→ **Proximal methods**  
[Combettes, Pesquet, 2011]

### Constrained approach

\[
\min_{f(x) \leq \eta} \|Ax - z\|^2
\]

→ **Inner iterations, ?**  
[Van Den Berg, Friedlander, 2008]
**Motivation**: Existing works – Poisson noise

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<td>( \min_{x \in \mathcal{H}} D_{KL}(Tx, z) + \lambda f(x) )</td>
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Motivation: Existing works – Poisson noise

Regularized approach

\[
\min_{x \in H} D_{KL}(Tx, z) + \lambda f(x)
\]

→ Cross-Entropy minimization
[Byrne, 1993]

→ Barrier function optimization
[Chouzenoux et al., 2011]

Constrained approach

\[
\min f(x) \quad \text{subject to} \quad D_{KL}(Tx, z) \leq \eta
\]

If \( f = \|F \cdot \|^2 \)

→ ?
**Motivation**: Existing works – Poisson noise

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Problem

\[
\hat{x} \in \text{Argmin}_{x \in \mathbb{R}^N} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} 
    h_1(H_1 x) \leq \eta_1, \\
    \vdots \\
    h_S(H_S x) \leq \eta_S, 
\end{cases}
\]

- (\forall s \in \{1, \ldots, S\}), \; H_r: \mathbb{R}^N \to \mathbb{R}^{M_s} \text{ is a linear operator},
- (\forall s \in \{1, \ldots, S\}), \; h_s \in \Gamma_0(\mathbb{R}^{M_s}),
- (\forall r \in \{1, \ldots, R\}), \; T_r: \mathbb{R}^N \to \mathbb{R}^{N_r} \text{ is a linear operator},
- (\forall r \in \{1, \ldots, R\}), \; g_r \in \Gamma_0(\mathbb{R}^{N_r}).

⇒ Any closed convex subset \( C_s \) of \( \mathbb{R}^{M_s} \) can be expressed in this way by setting \( \eta_s = 0, \; L = 1 \) and \( h_s = d_{C_s} = \| \cdot - P_{C_s} \| \)
Problem

\[ \hat{x} \in \mathop{\text{Argmin}}_{x \in \mathbb{R}^N} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases} \]

- (\forall s \in \{1, \ldots, S\}), \ H_r: \mathbb{R}^N \to \mathbb{R}^{M_s} \text{ is a bounded linear operator},
- (\forall s \in \{1, \ldots, S\}), \ C_s \text{ is a nonempty closed convex subset of } \mathbb{R}^{M_s},
- (\forall r \in \{1, \ldots, R\}), \ T_r: \mathbb{R}^N \to \mathbb{R}^{N_r} \text{ is a bounded linear operator},
- (\forall r \in \{1, \ldots, R\}), \ g_r \in \Gamma_0(\mathbb{R}^{N_r}).
Problem

\[ \hat{x} \in \text{Argmin} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases} \]

- **Forward-Backward** [Combettes, Wajs, 2005]

\[ \rightarrow \min_x g_1(T_1 x) + g_2(x) \text{ with } g_1 \text{ gradient Lipschitz function} \]
Motivation

Solution

Experiment 1

Experiment 2

Conclusions

Prox.

Problem

\[ \hat{x} \in \text{Argmin}_{x \in \mathbb{R}^N} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases} \]

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- **Douglas-Rachford**  [Combettes, Pesquet, 2007]
  \[ \rightarrow \min_x g_1(x) + g_2(x) \]
Problem

\[ \hat{x} \in \text{Argmin}_{x \in \mathbb{R}^N} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases} \]

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  \[ \rightarrow \min_x g_1(x) + g_2(x) \]

- **PPXA** [Combettes,Pesquet,2008]
  \[ \rightarrow \min_x \sum_{r=1}^{R} g_r(x) + \sum_{s=1}^{S} \iota_{C_s}(x) \]
Problem

\[ \hat{x} \in \text{Argmin} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ : \\ H_S x \in C_S, \end{cases} \]


\[ \rightarrow \min_x \sum_{r=1}^{R} g_r(T_r x) + \sum_{s=1}^{S} \iota_{C_s}(H_s x) \]

\[ \rightarrow \sum_{r=1}^{R} T_r^* T_r + \sum_{s=1}^{S} H_s^* H_s \text{ invertible} \]
Problem

\[ \hat{x} \in \text{Argmin}_{x \in \mathbb{R}^N} \sum_{r=1}^{R} g_r(T_rx) \quad \text{s.t.} \quad \begin{cases} H_1x \in C_1, \\ \vdots \\ H_Sx \in C_S, \end{cases} \]

- **PPXA +** [Pesquet,Pustelnik,2012] / **ADMM** [Setzer,Steidl,Teuber,2009]
  \[ \rightarrow \min_x \sum_{r=1}^{R} g_r(T_rx) + \sum_{s=1}^{S} \iota_{C_s}(H_sx) \]
  \[ \rightarrow \sum_{r=1}^{R} T_r^* T_r + \sum_{s=1}^{S} H_s^* H_s \text{ invertible} \]

- **M+SFBF** [Briceño-Arias,Combettes,2011]
  **M+LFBF** [Combettes,Pesquet,2012] and others [Vũ,2013][Condat,2013]
  \[ \rightarrow \min_x \sum_{r=1}^{R} g_r(T_rx) + \sum_{s=1}^{S} \iota_{C_s}(H_sx) \]
Problem

For \( n = 0, 1, \ldots \)

\[
\chi[n] = \sum_{r=1}^{R} \omega_r u_{r}^{[n]} + \sum_{s=1}^{S} \omega_s u_{s}^{[n]}
\]

For \( r = 1, \ldots, R \)

\[
w_{1,r}^{[n]} = u_{r}^{[n]} - \gamma \ell T_r v_{r}^{[n]}
\]
\[
w_{2,r}^{[n]} = v_{r}^{[n]} + \gamma \ell T_r u_{r}^{[n]}
\]

For \( s = 1, \ldots, S \)

\[
\bar{w}_{1,s}^{[n]} = u_{s}^{[n]} - \gamma_n H_s^* v_{s}^{[n]}
\]
\[
\bar{w}_{2,s}^{[n]} = u_{s}^{[n]} + \gamma_n H_s u_{s}^{[n]}
\]

\[
p_{1}^{[n]} = \sum_{r=1}^{R} \omega_r w_{1,r}^{[n]} + \sum_{s=1}^{S} \omega_s \bar{w}_{1,s}^{[n]}
\]

For \( r = 1, \ldots, R \)

\[
p_{2,r}^{[n]} = w_{2,r}^{[n]} - \frac{\gamma_n}{\omega_r} \text{prox}_{\omega_r \gamma_n \ell} \left( \frac{\omega_r w_{2,r}^{[n]}}{\gamma_n} \right)
\]

\[
q_{1,r}^{[n]} = p_{1}^{[n]} - \gamma_n (T_r^* p_{2,r}^{[n]})
\]
\[
q_{2,r}^{[n]} = p_{2,r}^{[n]} + \gamma_n (T_r p_{1}^{[n]})
\]

Update \( u_{1}^{[n+1]} \) and \( v_{1}^{[n+1]} \)

For \( s = 1, \ldots, S \)

\[
p_{2,s}^{[n]} = \bar{w}_{2,r}^{[n]} - \frac{\gamma_n}{\omega_s} P_{C_s} \left( \frac{\omega_s \bar{w}_{2,r}^{[n]}}{\gamma_n} \right)
\]

\[
\bar{q}_{1,s}^{[n]} = p_{1}^{[n]} - \gamma_n (H_s^* p_{2,s}^{[n]})
\]
\[
\bar{q}_{2,s}^{[n]} = p_{2,s}^{[n]} + \gamma_n (H_s p_{1}^{[n]})
\]

Update \( u_{1}^{[n+1]} \) and \( v_{1}^{[n+1]} \)

Under technical assumptions, \( (\chi^{[n]})_{n \in \mathbb{N}} \) generated by M+SFBF [Combettes, Briceño-Arias, 2011] converges to \( \hat{x} \)

Proximity operator computation

Projection computation
Definition [Moreau, 1965] Let $f \in \Gamma_0(H)$ where $H$ denotes a real Hilbert space. The proximity operator of $f$ at point $u \in H$ is the unique point denoted by $\text{prox}_f u$ such that

\[
(\forall u \in H) \quad \text{prox}_f u = \arg \min_{v \in H} f(v) + \frac{1}{2} \|u - v\|^2
\]
Proximity operator

**Definition** [Moreau, 1965] Let \( f \in \Gamma_0(\mathcal{H}) \) where \( \mathcal{H} \) denotes a real Hilbert space. The proximity operator of \( f \) at point \( u \in \mathcal{H} \) is the unique point denoted by \( \text{prox}_f u \) such that

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\]

**Examples:** closed form

- \( \text{prox}_{\chi \| \cdot \|_1} \): soft-thresholding with a fixed threshold \( \chi > 0 \)
Proximity operator

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$$(\forall u \in \mathcal{H}) \quad \text{prox}_f u = \arg \min_{v \in \mathcal{H}} f(v) + \frac{1}{2} \|u - v\|^2$$

**Examples:**

- $\text{prox}_{\chi \| \cdot \|_1}$: soft-thresholding with a fixed threshold $\chi > 0$
- $\text{prox}_{\| \cdot \|_{1,2}}$ [Peyré, Fadili, 2011].
- $\text{prox}_{D_{KL}}$ [Chaux, Combettes, Pesquet, Wajs, 2005].
- $\text{prox}_{\iota_C} = P_C$: projection onto the convex set $C$. 
Definition [Moreau,1965] Let \( f \in \Gamma_0(\mathcal{H}) \) where \( \mathcal{H} \) denotes a real Hilbert space. The proximity operator of \( f \) at point \( u \in \mathcal{H} \) is the unique point denoted by \( \text{prox}_f u \) such that

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- \( \text{prox}_{D_{KL}} \) [Chaux,Combettes,Pesquet,Wajs,2005].
- \( \text{prox}_{\nu_C} = P_C \): projection onto the convex set \( C \).
  - range constraint: hypercube projection,
  - closed half-space: half-space projection,
Proximity operator

**Definition** [Moreau, 1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of $f$ at point $u \in \mathcal{H}$ is the unique point denoted by $\text{prox}_f u$ such that \[(\forall u \in \mathcal{H}) \quad \text{prox}_f u = \arg \min_{v \in \mathcal{H}} f(v) + \frac{1}{2} \| u - v \|^2 \]

**Examples:** NO closed form

- $\text{prox}_C = P_C$: projection onto the convex set $C$.
  - $C$ models a $\ell_{1,p}$-ball constraint: iterative procedure for projection [Quattoni, Carreras, Collins, Darrell, 2007] [Van Den Berg, Friedlander, 2008].
  - constraint associated with the Kullback-Leibler divergence
  - constraint associated with the logistic cost function
Solution

Assumption: separable function

For every $y = [(y^{(1)})^\top, \ldots, (y^{(L)})^\top]^\top \in \mathbb{R}^M$,

$$y \in C \iff h(y) \leq \eta \iff \sum_{\ell=1}^{L} h^{(\ell)}(y^{(\ell)}) \leq \eta.$$
Solution

- **Assumption:** separable function
  
  For every \( y = [(y^{(1)})^\top, \ldots, (y^{(L)})^\top]^\top \in \mathbb{R}^M \),
  
  \[
  y \in C \iff h(y) \leq \eta \iff \sum_{\ell=1}^{L} h^{(\ell)}(y^{(\ell)}) \leq \eta. 
  \]

- **Solution:** splitting the constraint into simpler constraints by introducing the auxiliary vector \( \zeta = (\zeta^{(\ell)})_{1 \leq \ell \leq L} \in \mathbb{R}^L \),

  \[
  y \in C \iff \left\{ \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta, \quad (\forall \ell \in \{1, \ldots, L\}) \right\} h^{(\ell)}(y^{(\ell)}) \leq \zeta^{(\ell)}. 
  \]
Solution

- **Assumption:** separable function
  
  For every \( y = [(y^{(1)})^\top, \ldots, (y^{(L)})^\top]^\top \in \mathbb{R}^M, \)

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  \[
  y \in C \iff \left\{ \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta, \quad (\forall \ell \in \{1, \ldots, L\}) \left( y^{(\ell)}, \zeta^{(\ell)} \right) \in \text{epi} h^{(\ell)} \right\}.
  \]
Solution

\( y \in C \iff \left\{ \begin{array}{l}
\zeta \in V \\
(y, \zeta) \in E
\end{array} \right. \)

- \( V \) denotes a closed half-space such that:

\[
V = \{ \zeta \in \mathbb{R}^L \mid 1_L^T \zeta \leq \eta \}
\]

- \( E \) is the closed convex set associated to the epigraphical constraint:

\[
E = \{ (y, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\}) \ (y^{(\ell)}, \zeta^{(\ell)}) \in \text{epi} \ h^{(\ell)} \}
\]
Solution

\[ y \in C \iff \begin{cases} \zeta \in V \\ (y, \zeta) \in E \end{cases} \]

- \( V \) denotes a closed half-space such that:

\[ V = \{ \zeta \in \mathbb{R}^L \mid 1^T_L \zeta \leq \eta \} \]

\( \rightarrow P_V \) has a closed form: projection onto an half-space.

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Solution

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\[ E = \{(y, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\}) \ (y^{(\ell)}, \zeta^{(\ell)}) \in \text{epi} \ h^{(\ell)} \} \]

\( \rightarrow P_E \) has a closed form for specific choice of \( h^{(\ell)} \).
Solution

- **Euclidean norm** functions defined as:

\[
(\forall \ell \in \{1, \ldots, L\}) (\forall y^{(\ell)} \in \mathbb{R}^{M^{(\ell)}}) \quad h^{(\ell)}(y^{(\ell)}) = \tau^{(\ell)} \| y^{(\ell)} \|
\]

where \( \tau^{(\ell)} \in ]0, +\infty[. \)
Euclidean norm functions defined as:

\[(\forall \ell \in \{1, \ldots, L\}) (\forall y^{(\ell)} \in \mathbb{R}^{M^{(\ell)}}) \quad h^{(\ell)}(y^{(\ell)}) = \tau^{(\ell)} \| y^{(\ell)} \|\]

where \(\tau^{(\ell)} \in ]0, +\infty[\).

Epigraphic projection: for every \((y^{(\ell)}, \zeta^{(\ell)}) \in \mathbb{R}^{M^{(\ell)}} \times \mathbb{R}\)

\[P_{\text{epi} h^{(\ell)}}(y^{(\ell)}, \zeta^{(\ell)}) = \begin{cases} 
(y^{(\ell)}, \zeta^{(\ell)}), & \text{if } \| y^{(\ell)} \| < \frac{\zeta^{(\ell)}}{\tau^{(\ell)}}, \\
(0, 0), & \text{if } \| y^{(\ell)} \| < -\tau^{(\ell)} \zeta^{(\ell)}, \\
\alpha^{(\ell)} (y^{(\ell)}, \tau^{(\ell)} \| y^{(\ell)} \|), & \text{otherwise},
\end{cases}\]

where \(\alpha^{(\ell)} = \frac{1}{1 + (\tau^{(\ell)})^2} \left(1 + \frac{\tau^{(\ell)} \zeta^{(\ell)}}{\| y^{(\ell)} \|}\right)\).
Solution

- **Infinity norms** defined as:

\[
\left( \forall \ell \in \{1, \ldots, L\} \right) \left( \forall y^{(\ell)} = (y^{(\ell,m)})_{1 \leq m \leq M^{(\ell)}} \in \mathbb{R}^{M^{(\ell)}} \right)
\]

\[
h^{(\ell)}(y^{(\ell)}) = \max \left\{ \frac{|y^{(\ell,m)}|}{\tau^{(\ell,m)}} \mid 1 \leq m \leq M^{(\ell)} \right\}
\]

where \((\tau^{(\ell,m)})_{1 \leq m \leq M^{(\ell)}} \in ]0, +\infty[^{M^{(\ell)}}\).
Solution

- **Epigraphic projection:**
  - \((\nu^{(\ell,m)})_{1 \leq m \leq M^{(\ell)}}\): sequence of reals by sorting \((|y^{(\ell,m)}|/\tau^{(\ell,m)})_{1 \leq m \leq M^{(\ell)}}\)
in ascending order \((\nu^{(\ell,0)} = -\infty\) and \(\nu^{(\ell,M^{(\ell)}+1)} = +\infty\)).
  - \(\overline{m}\) is the unique integer in \(\{1, \ldots, M^{(\ell)} + 1\}\) such that
    \[
    \nu^{(\ell,\overline{m}-1)} < \frac{\zeta^{(\ell)} + \sum_{m=\overline{m}}^{M^{(\ell)}} \nu^{(\ell,m)}(\tau^{(\ell,m)})^2}{1 + \sum_{m=\overline{m}}^{M^{(\ell)}}(\tau^{(\ell,m)})^2} \leq \nu^{(\ell,\overline{m})}.
    \]
  - \((p^{(\ell)}, \theta^{(\ell)}) = P_{\text{epi}} h^{(\ell)}(y^{(\ell)}, \zeta^{(\ell)})\) with \(p^{(\ell)} = (p^{(\ell,m)})_{1 \leq m \leq M^{(\ell)}}\),

where

\[
p^{(\ell,m)} = \begin{cases} 
    y^{(\ell,m)}, & \text{if } |y^{(\ell,m)}| \leq \tau^{(\ell,m)}\theta^{(\ell)}, \\
    \tau^{(\ell,m)}\theta^{(\ell)}, & \text{if } y^{(\ell,m)} > \tau^{(\ell,m)}\theta^{(\ell)}, \\
    -\tau^{(\ell,m)}\theta^{(\ell)}, & \text{if } y^{(\ell,m)} < -\tau^{(\ell,m)}\theta^{(\ell)},
\end{cases}
\]

and

\[
\theta^{(\ell)} = \max \left( \zeta^{(\ell)} + \sum_{m=\overline{m}}^{M^{(\ell)}} \nu^{(\ell,m)}(\tau^{(\ell,m)})^2, 0 \right) \\
1 + \sum_{m=\overline{m}}^{M^{(\ell)}}(\tau^{(\ell,m)})^2.
\]
RGB image restoration with missing samples

- Original (multicomponent) image: \( \overset{\_}{x} = (\overset{\_}{x}_1, \ldots, \overset{\_}{x}_R) \in (\mathbb{R}^M)^R \)
- Linear operator: \( A = (A_{j,i})_{1 \leq j \leq S, 1 \leq i \leq R}, \text{ with } A_{j,i} \in \mathbb{R}^{K \times M} \)
- Zero-mean white Gaussian noise: \( w \in (\mathbb{R}^K)^S \)
- Degraded image: \( z = (z_1, \ldots, z_S) \in (\mathbb{R}^K)^S \)

\[
\overset{\_}{x} \xrightarrow{z} z = A\overset{\_}{x} + w
\]
RGB image restoration with missing samples

\[
\hat{x} \in \text{Argmin}_{x \in (\mathbb{R}^M)^R} \left\{ \|Ax - z\|_2^2 + \lambda g(x) \right\} 
\]

- **Component-wise Total Variation (CC-TV)**
  - [Blomgren 1998] [Zach 2007]

- **Structure Tensor TV (ST-TV)**
  - \( \ell_p \) matrix-norm regularization
  - [Di Zenzo 1986] [Sapiro 1996] [Weickert 1999] [Tschumperlé 2001]
  - [Bresson 2008] [Duval 2009] [Goldluecke 2012]
RGB image restoration with missing samples

\[ \hat{x} \in \text{Argmin} \|Ax - z\|_2^2 \quad \text{subj. to} \quad g(x) \leq \eta \]

- Constrained approach
- Regularization by ST Non-Local TV (ST-NLTV)
  - NLTV better preserves texture, details and fine structures
  - ST better reveals features not visible in single components
RGB image restoration with missing samples

1. Non-Local gradient at point $\ell \in \{1, \ldots, M\}$
   \[
   X^{(\ell)} = \left( \omega_{\ell,n} \left( x_i^{(\ell)} - x_i^{(n)} \right) \right)_{n \in \mathcal{N}_\ell, 1 \leq i \leq R} \in \mathbb{R}^{M_\ell \times R}
   \]

2. ST-NLTV
   \[
   g(x) = \sum_{\ell=1}^{M} \tau_\ell \|X^{(\ell)}\|_p \iff g(x) = \sum_{\ell=1}^{M} \tau_\ell \left( \min\{M_\ell,R\} \sum_{m=1}^{\min\{M_\ell,R\}} \left( \sigma_{X^{(\ell)}}^{(m)} \right)^p \right)^{1/p}
   \]
RGB image restoration with missing samples

- Non-Local gradient at point $\ell \in \{1, \ldots, M\}$
  
  \[
  X^{(\ell)} = \left( \omega_{\ell,n} \left( x_i^{(\ell)} - x_i^{(n)} \right) \right)_{n \in \mathcal{N}_\ell, 1 \leq i \leq R} \in \mathbb{R}^{M_\ell \times R}
  \]

- Special case: ST-TV
  - $\mathcal{N}_\ell \rightarrow$ horizontal/vertical neighbours
  - $\omega_{\ell,n} \equiv 1$
RGB image restoration with missing samples

\[ \hat{x} \in \text{Argmin}_{x \in C} \| Ax - z \|_2^2 \quad \text{subject to} \quad Fx \in D \]
RGB image restoration with missing samples

\[
\widehat{x} \in \text{Argmin}_{x \in \mathcal{C}} \|Ax - z\|_2^2 \quad \text{subject to} \quad Fx \in D
\]

\[
\vdots
\]

\[
(\widehat{x}, \widehat{\zeta}) \in \text{Argmin}_{(x, \zeta) \in \mathcal{C} \times \mathcal{V}} \|Ax - z\|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E
\]
RGB image restoration with missing samples

\[ \hat{x} \in \text{Argmin} \| Ax - z \|_2^2 \quad \text{subject to} \quad Fx \in D \]

\[ (\hat{x}, \hat{\zeta}) \in \text{Argmin} \| Ax - z \|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E \]

- Collection of epigraphs

\[ E = \{(X, \zeta) \mid (X^{(\ell)}, \zeta^{(\ell)}) \in \text{epi} \| \cdot \|_p \quad (\forall \ell \in \{1, \ldots, M\})\} \]
RGB image restoration with missing samples

\[ \hat{x} \in \text{Argmin} \| Ax - z \|_2^2 \quad \text{subject to} \quad Fx \in D \]

\[ \vdots \]

\[ (\hat{x}, \hat{\zeta}) \in \text{Argmin} \| Ax - z \|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E \]

Collection of epigraphs

\[ E = \left\{ (X, \zeta) \mid (X^{(\ell)}, \zeta^{(\ell)}) \in \text{epi} \| \cdot \|_p \quad (\forall \ell \in \{1, \ldots, M\}) \right\} \]

Closed half-space

\[ V = \left\{ \zeta \in \mathbb{R}^M \mid 1_M^T \zeta \leq \eta \right\} \]

with \( 1_M = (1, \ldots, 1)^T \in \mathbb{R}^M \)
RGB image restoration with missing samples

- $P_{\text{epi}} \parallel \cdot \parallel_p$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
RGB image restoration with missing samples

- $P_{\text{epi}} \parallel \cdot \parallel_p$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  - [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  - → can we extend these results to matrix norms?
RGB image restoration with missing samples

- $P_{\text{epi}} \parallel \cdot \parallel_p$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  - [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  - can we extend these results to matrix norms?

- S.V.D. : $X^{(\ell)} = U^{(\ell)} \text{Diag}(s^{(\ell)}) V^{(\ell)^\top}$
RGB image restoration with missing samples

- $P_{\text{epi}}\|\cdot\|_p$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  - [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  - → can we extend these results to matrix norms?
- S.V.D. : $X^{(\ell)} = U^{(\ell)} \text{Diag}(s^{(\ell)}) \ V^{(\ell)^\top}$
- Proximity operator of spectral functions [Lewis 1995]

\[
\text{prox}_{\|\cdot\|_p}(X^{(\ell)}) = U^{(\ell)} \text{Diag}(\text{prox}_{\|\cdot\|_p}(s^{(\ell)})) \ V^{(\ell)^\top}
\]
RGB image restoration with missing samples

- $P_{\text{epi}}_{\cdot \cdot \cdot p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  - [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  - can we extend these results to matrix norms?
- S.V.D. : $X^{(\ell)} = U^{(\ell)} \text{ Diag}(s^{(\ell)}) \ V^{(\ell)\top}$
- Proximity operator of spectral functions [Lewis 1995]
  - \[ \text{prox}_{\cdot \cdot \cdot p}(X^{(\ell)}) = U^{(\ell)} \ \text{Diag}(\text{prox}_{\cdot \cdot \cdot p}(s^{(\ell)})) \ V^{(\ell)\top} \]

Epigraphical projection

1. $P_{\text{epi}}_{\cdot \cdot \cdot p}(X^{(\ell)}, \zeta^{(\ell)}) = \left( U^{(\ell)} \ \text{Diag}(t^{(\ell)}) \ V^{(\ell)\top}, \ \theta^{(\ell)} \right)$
RGB image restoration with missing samples

- $P_{\text{epi}} \| \cdot \|_p$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  - [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  - → can we extend these results to matrix norms?
- S.V.D. : $X^{(\ell)} = U^{(\ell)} \text{Diag}(s^{(\ell)}) \ V^{(\ell)\top}$
- Proximity operator of spectral functions [Lewis 1995]

$$\text{prox}_{\| \cdot \|_p}(X^{(\ell)}) = U^{(\ell)} \text{Diag}(\text{prox}_{\| \cdot \|_p}(s^{(\ell)})) \ V^{(\ell)\top}$$

Epigraphical projection

1. $P_{\text{epi}} \| \cdot \|_p(X^{(\ell)}, \zeta^{(\ell)}) = \left(U^{(\ell)} \text{Diag}(t^{(\ell)}) \ V^{(\ell)\top}, \theta^{(\ell)}\right)$
2. $(t^{(\ell)}, \theta^{(\ell)}) = P_{\text{epi}} \| \cdot \|_p(s^{(\ell)}, \zeta^{(\ell)})$
RGB image restoration with missing samples

\[(\hat{x}, \hat{\zeta}) \in \text{Argmin} \|Ax - z\|^2_2 \text{ subject to } (Fx, \zeta) \in E\]

- Degradation: 3 × 3 uniform blur, 90% of decimation, AWGN with \( \alpha = 10 \)

- Color space: RGB
  - pixels of \( z \) have missing colors
  - impossible to work into YCbCr, CIELab, ...

- Dynamics range constraint: \( x_{i}^{(\ell)} \in [0, 255] \)

- Weights \( \omega_{\ell,n} \) estimated as in [Foi 2012]

- Choice of \( \eta \) based on image characteristics
RGB image restoration with missing samples

Original

Noisy

Zoom
RGB image restoration with missing samples

$\ell_1$-CC-TV
16.15 dB

$\ell_2$-CC-TV
16.32 dB

$\ell_\infty$-CC-TV
16.05 dB
RGB image restoration with missing samples

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RGB image restoration with missing samples

\[ \ell_1\text{-ST-NLTV} \quad 18.20 \text{ dB} \]
\[ \ell_2\text{-ST-NLTV} \quad 17.46 \text{ dB} \]
\[ \ell_\infty\text{-ST-NLTV} \quad 16.67 \]
RGB image restoration with missing samples
Poisson based restoration

\[
\begin{align*}
\text{minimize} & \quad \sum_{x \in \mathbb{R}^N} \left\| B_b Fx \right\|_{L_1} \\
\text{subject to} & \quad \begin{cases} 
    x \in C \\
    g(Ax, z) \leq \eta.
\end{cases}
\end{align*}
\]

- For computational reasons, it will be assumed that there exists a partition of \( \mathbb{I} \) in \( S \) subsets \((\mathbb{I}_s)_{1 \leq s \leq S}\) such that
  \[
  \sum_{b \in \mathbb{I}} \| B_b \cdot \| = \sum_{s=1}^{S} \sum_{b \in \mathbb{I}_s} \| B_b \cdot \| \quad \text{(i.e. grouped into } S \text{ sets of non-overlapping blocks)}.
  \]

- \textbf{Particular case} : \( S = 1 \), \( \mathbb{I} = \mathbb{I}_1 = K \) and, for every \( b \in \mathbb{I} \), \( B_b \) selects one element (i.e. one pixel) \( \rightarrow \text{the classical } \ell^1\text{-norm is obtained.} \)
**Poisson based restoration**

minimize\[
\sum_{s=1}^{S} \sum_{b \in \mathbb{L}_s} \| B_b Fx \| \quad \text{subject to} \quad \begin{cases} 
\mathcal{X} \in C \\
g(Ax, z) \leq \eta.
\end{cases}
\]

- For computational reasons, it will be assumed that there exists a partition of \(\mathbb{L}\) in \(S\) subsets \((\mathbb{L}_s)_{1 \leq s \leq S}\) such that \(\sum_{b \in \mathbb{L}} \| B_b \cdot \| = \sum_{s=1}^{S} \sum_{b \in \mathbb{L}_s} \| B_b \cdot \|\) (i.e. grouped into \(S\) sets of non-overlapping blocks).

- **Particular case**: \(S = 1\), \(\mathbb{L} = \mathbb{L}_1 = \mathbb{K}\) and, for every \(b \in \mathbb{L}\), \(B_b\) selects one element (i.e. one pixel) \(\rightarrow\) the classical \(\ell^1\)-norm is obtained.
Poisson based restoration

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} \sum_{b \in \mathbb{L}_s} \|B_b Fx\| \\
\text{subject to} & \quad \begin{cases} 
  x \in C \\
  g(Ax, z) \leq \eta.
\end{cases}
\end{align*}
\]

that is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} \sum_{b \in \mathbb{L}_s} \|B_b Fx\| \\
\text{subject to} & \quad \begin{cases} 
  x \in C \\
  Ax \in D
\end{cases}
\end{align*}
\]

with \( D = \{ u \in \mathbb{R}^K \mid g(u, z) \leq \eta \} = \text{lev}_{\leq \eta} g(\cdot, z) \).
Poisson based restoration

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} \sum_{b \in L_s} \| B_b Fx \| \quad \text{subject to} \quad \begin{cases} 
x \in C \\
g(Ax, z) \leq \eta. \end{cases}
\end{align*}
\]

that is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} \sum_{b \in L_s} \| B_b Fx \| \quad \text{subject to} \quad \begin{cases} 
x \in C \\
Ax \in D 
\end{cases}
\end{align*}
\]

with \( D = \{ u \in \mathbb{R}^K \mid g(u, z) \leq \eta \} = \lev_{\leq \eta} g(\cdot, z). \)

- **Projection onto** \( D \)
  -> Closed form if \( g(\cdot, z) = \| \cdot - z \|^2 \) [Rockafellar, 1969].
  -> **NO closed form in a general context.**
Poisson based restoration

Explicit form of the projection operator associated with:

\[ h_\ell(v_\ell) = \max\{v_\ell(j) + \eta_\ell(j) \mid 1 \leq j \leq M_\ell\} \]

where

\[ \rightarrow \quad v_\ell = (v_{\ell,1}, \ldots, v_{\ell,M_\ell})^\top \in \mathbb{R}^{M_\ell} \]

\[ \rightarrow \quad \ell \in \{1, \ldots, L\} \text{ and } (\eta_{\ell,1}, \ldots, \eta_{\ell,M_\ell})^\top \in \mathbb{R}^{M_\ell} \]

Example for \( L = 1 \) and \( M^{(1)} = 3 \):

![Graphical representation of the projection operator](image-url)
Explicit form of the projection operator associated with:

\[
h_\ell(v^{\ell}) = \max\{v^{(\ell,j)} + \eta^{(\ell,j)} | 1 \leq j \leq M^{(\ell)}\}
\]

where

\[
\rightarrow v^{(\ell)} = (v^{(\ell,1)}, \ldots, v^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}} \\
\rightarrow \ell \in \{1, \ldots, L\} \text{ and } (\eta^{(\ell,1)}, \ldots, \eta^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}}
\]

Example for \(L = 1\) and \(M^{(1)} = 3\):

![Graph showing an example for Poisson based restoration](image)
Poisson based restoration

\[
g(u, z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \approx \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)})
\]

- \( h_\ell(v^{(\ell)}) = \max\{v^{(\ell,j)} + \eta^{(\ell,j)} | 1 \leq j \leq M^{(\ell)}\}, \)
- \( \eta^{(\ell,j)} = g_\ell(a_j^{(\ell)}, z^{(\ell)}) - \delta_j^{(\ell)} a_j^{(\ell)}, \)
- \( \delta_j^{(\ell)} \in \mathbb{R} \) is any subgradient of \( g_r(\cdot, z^{(\ell)}) \) at \( a_j^{(\ell)}, \)
- \( \Delta^{(\ell)} = [\delta_1^{(\ell)}, \ldots, \delta_{M^{(\ell)}}^{(\ell)}]^T. \)

→ The approximation can be as close as desired by choosing \( M^{(\ell)} \) large enough.
Poisson based restoration

\[ g(u, z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \approx \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)}) \]

- \( h_\ell(v^{(\ell)}) = \max\{v^{(\ell,j)} + \eta^{(\ell,j)} | 1 \leq j \leq M^{(\ell)}\} \),
- \( \eta^{(\ell,j)} = g_\ell(a^{(\ell)}_j, z^{(\ell)}) - \delta^{(\ell)}_j a^{(\ell)}_j \),
- \( \delta^{(\ell)}_j \in \mathbb{R} \) is any subgradient of \( g_r(\cdot, z^{(\ell)}) \) at \( a^{(\ell)}_j \),
- \( \Delta^{(\ell)} = [\delta^{(\ell)}_1, \ldots, \delta^{(\ell)}_{M^{(\ell)}}]^\top \).

\[ \rightarrow \] The approximation can be as close as desired by choosing \( M^{(\ell)} \) large enough.
Poisson based restoration

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} \sum_{b \in L_s} \| B_b Fx \| \\
\text{subject to} & \quad \begin{cases} 
  x \in C \\
  Ax \in D
\end{cases}
\end{align*}
\]

⇒ Approximated criterion :

\[
\begin{align*}
\text{minimize} & \quad \sum_{(x, \zeta) \in \mathbb{R}^N \times \mathbb{R}^L} \sum_{s=1}^{S} \sum_{b \in L_s} \| B_b Fx \| \\
\text{subject to} & \quad \begin{cases} 
  (x, \zeta) \in C \times V \\
  \Delta Ax \in E
\end{cases}
\end{align*}
\]

where

- \( D = \{ u \in \mathbb{R}^L \mid g(u, z) \leq \eta \} \),
- \( V = \{ \zeta \in \mathbb{R}^L \mid 1_L^\top \zeta \leq \eta \} \),
- \( E = \{ (v, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\}) (v^{(\ell)}, \zeta^{(\ell)}) \in \text{epi } h_\ell \} \),
- For every \( u \in \mathbb{R}^L \), \( g(u, z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \approx \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)}) \).
Poisson based restoration

- Electron microscopy image of size $\bar{N} = 128 \times 128$,
- $T$ denotes a randomly decimated blur: uniform blur of size $3 \times 3$ and approximately 60% of missing data, that leads to $L = 9834$,
- Poisson noise with scaling parameter 0.5.
Poisson based restoration

Choice of the criterion:
- Data fidelity: approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
  - $C = [0, 255]^N$,
- $F$: Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks:
  - $\ell_1$-reg: Classical $\ell_1$ cost function,
Poisson based restoration

**Choice of the criterion:**
- Data fidelity: approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^N$,
- $F$: Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
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Poisson based restoration

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- Blocks:
  - $\ell_1$-reg: Classical $\ell_1$ cost function,
  - Block_PrimalDual: Blocks gathering primal and dual DTT coefficients,
### Poisson based restoration

**Choice of the criterion**:
- Data fidelity: approximation of the Poisson likelihood,
  - Influence of \( M \equiv M(\ell) \),
  - \( C = [0, 255]^{\mathbb{N}} \),
- \( F \): Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks:
  - \( \ell_1\)-reg: Classical \( \ell_1 \) cost function,
  - Block\_Primal\_Dual: Blocks gathering primal and dual DTT coefficients,
  - Block\_4Pixel\_overlap: spatially overlapping blocks of size \( 2 \times 2 \) are employed for each tree (primal or dual) separately.
Poisson based restoration

**Choice of the criterion:**

- Data fidelity: approximation of the Poisson likelihood,
  - Influence of $M \equiv M(\ell)$,
  - $C = [0, 255]^N$,
- $F$: Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks:
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Poisson based restoration

Choice of the criterion:
- Data fidelity: approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
  - $C = \mathbb{R}^N$,
- $F$: Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks:
  - $\ell_1$-reg: Classical $\ell_1$ cost function,
  - Block_PrimalDual: Blocks gathering primal and dual DTT coefficients,
  - Block_4Pixel_overlap: spatially overlapping blocks of size $2 \times 2$ are employed for each tree (primal or dual) separately.
Poisson based restoration

- Impact of $M$ and of the regularization term.
Poisson based restoration

- \( M = 7, \)
- Impact of the regularization term.

\( \ell_1 \)-reg
SNR = 16.3 dB

Block_PrimalDual
SNR = 16.5 dB

Block_4Pixel_overlap
SNR = 16.6 dB
Conclusions

\[
\text{Argmin} \sum_{r=1}^{R} g_r(T_rx) \quad \text{s.t.} \quad \begin{cases} 
\sum_{\ell=1}^{L} h_1^{(\ell)} ((H_1x)^{(\ell)}) \leq \eta_1 \\
H_2x \in C_2 \\
\vdots \\
H_Sx \in C_S
\end{cases}
\]

\[
\text{Argmin} \sum_{\ell=1}^{R} g_r(T_rx) \quad \text{s.t.} \quad \begin{cases} 
(\forall \ell \in \{1, \ldots, L\}) \quad h_1^{(\ell)} ((H_1x)^{(\ell)}) \leq \zeta^{(\ell)} \\
\sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta_1 \\
H_2x \in C_2 \\
\vdots \\
H_Sx \in C_S
\end{cases}
\]
Conclusions

\[
\begin{align*}
\text{Argmin}_{x} & \quad \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \sum_{\ell=1}^{L} h_1^{(\ell)}(\{H_1 x\}^{(\ell)}) \leq \eta_1 \\
& \quad H_2 x \in C_2 \\
& \quad \vdots \\
& \quad H_S x \in C_S
\end{align*}
\]

\[
\begin{align*}
\text{Argmin}_{x, \zeta} & \quad \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad (\forall \ell \in \{1, \ldots, L\}) \quad h_1^{(\ell)}(\{H_1 x\}^{(\ell)}) \leq \zeta^{(\ell)} \\
& \quad \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta_1 \\
& \quad H_2 x \in C_2 \\
& \quad \vdots \\
& \quad H_S x \in C_S
\end{align*}
\]

\[\rightarrow P_{\text{epi } h_1^{(\ell)}}: \text{ closed form when } h_1^{(\ell)} \text{ models a Euclidean or infinity norm.}\]
Conclusions

- **ℓ₁-ST-TV**
- **ℓ₁-ST-NLTV**
Conclusions

→ Faster than direct methods
  [Quattoni, Carreras, Collins, Darrell, 2007]  [Van Den Berg, Friedlander, 2008].
→ Links with bundle methods?
Conclusions

\[ h_1 \circ \Delta^{(1)} \]

→ Links with bundle methods?
References

