Accelerating Benders decomposition by level stabilization

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Outline

Outline

Generalized Benders decomposition

The problem

$$\begin{array}{ll} \min & f(x, y) \\ \text{s.t.} & G(x, y) \leq 0 \\ & x \in X,\ y \in Y \end{array}$$

where y is a complicating variable

The idea: variable decomposition [Geoffrion 1972]

$$\min_{y \in Y} v(y) \quad \text{with} \quad v(y) := \min_{x \in X, G(x,y) \leq 0} f(x, y)$$

$$v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

Example: network design problems

Fixed charged uncapicitated network design

- Variables
  - x : flow quantity
  - y : network links
- Formulation

$$
\begin{aligned}
\min \quad & \sum_{ijk} c_{ijk} x_{ijk} + \sum_{ij} f_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{j} x_{ijk} - \sum_{j} x_{jik} = \begin{cases} d_k & i = O(k) \\ -d_k & i = D(k) \\ 0 & \text{otherwise} \end{cases} \\
& x_{ijk} \leq d_k y_{ij} \\
& x \geq 0, \quad y \in \{0,1\}^{n^2}
\end{aligned}
$$

Fixing y leads to K independent network flow subproblems

General case of Benders decomposition

Assumptions

1. v **is convex** (and we can compute a subgradient)

$$v(y) := \min_{x \in X, G(x,y) \leq 0} f(x, y)$$

2. There is **no duality gap** for the subproblem

$$v(y) = \max_{u \geq 0} \theta_y(u) = \max_{u \geq 0} \min_{x \in X} \{ f(x, y) + u^\top G(x, y) \}$$

Benders master problem

- Cutting-plane model

$$\begin{aligned} \check{v}_k(y) &:= \max_{i \in I_k} \{ v(y_i) + g_i^\top (y - y_i) \} &\leq v(y) \\ \check{V}_k &:= \{ y \in Y : \beta_j^\top y \leq \alpha_j,\ j \in J_k \} &\supseteq \operatorname{dom} v \end{aligned}$$

- Benders **relaxed master**: (MILP)

$$y_{k+1} \in \operatorname{argmin} \check{v}_k(y) \quad \text{s.t.} \quad y \in \check{V}_k$$

Benders method

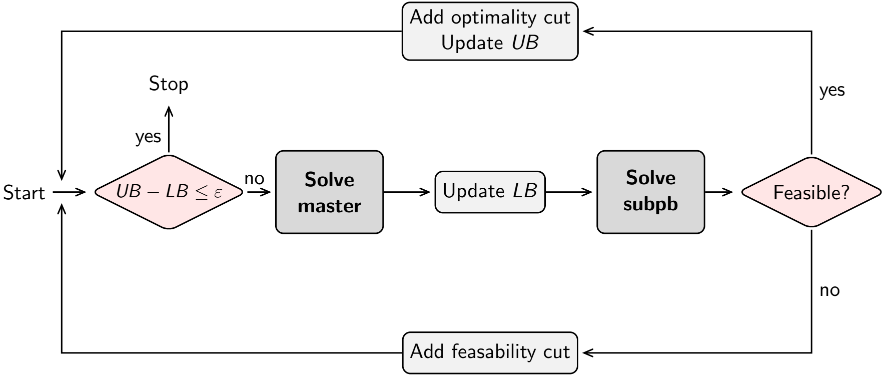
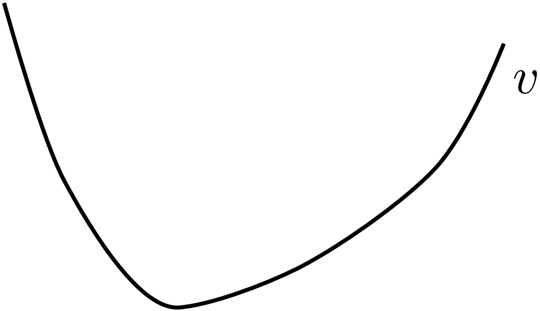


Illustration of Benders method



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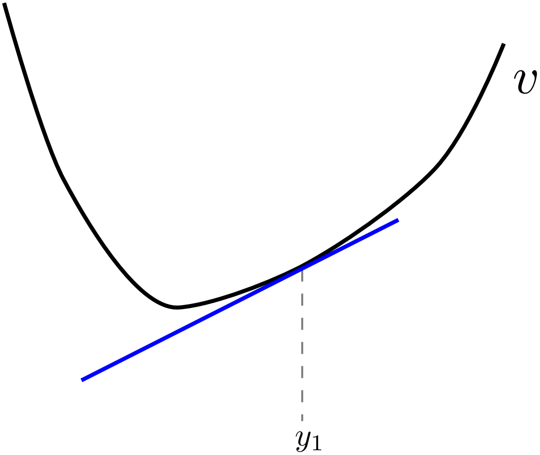

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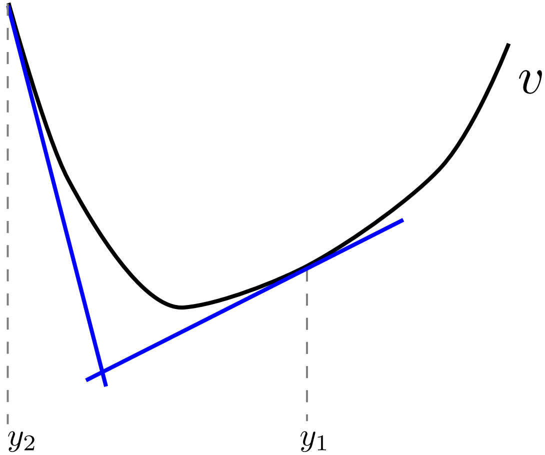

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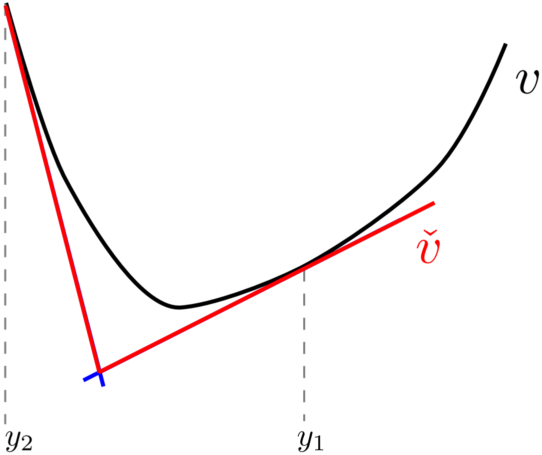

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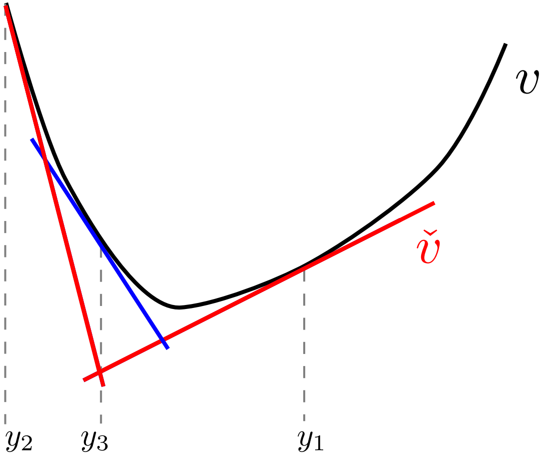

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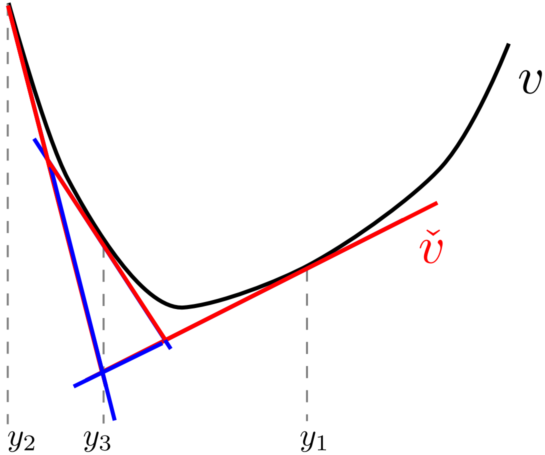

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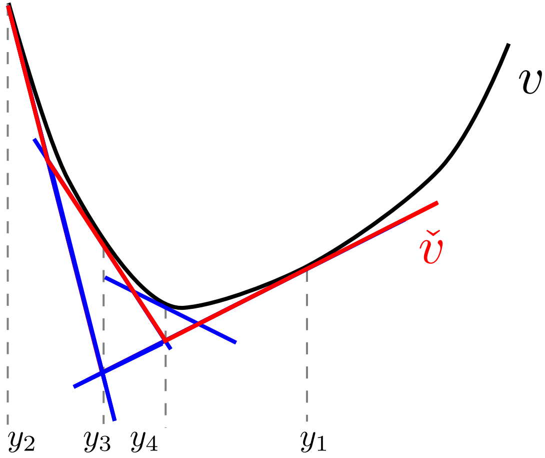

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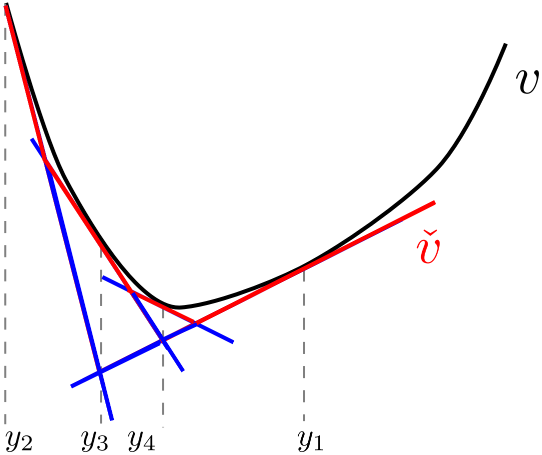

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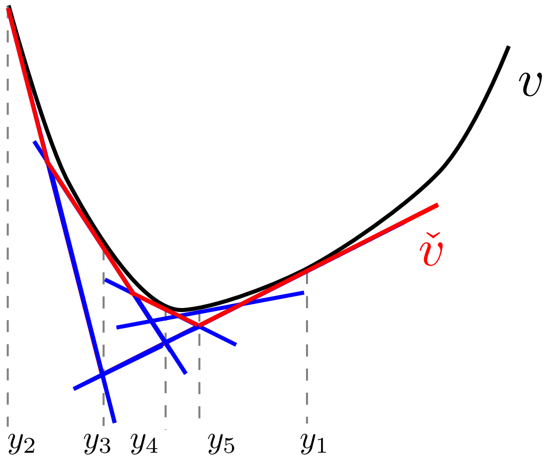

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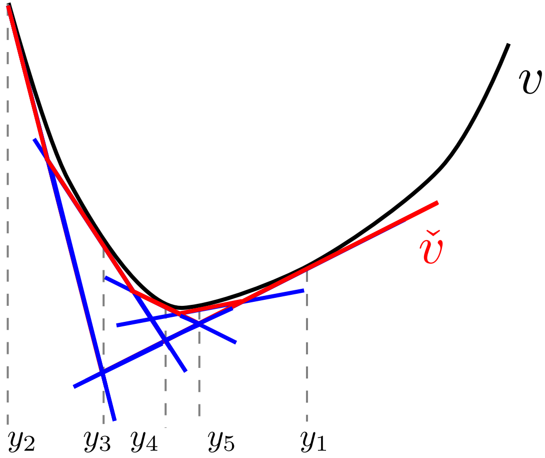

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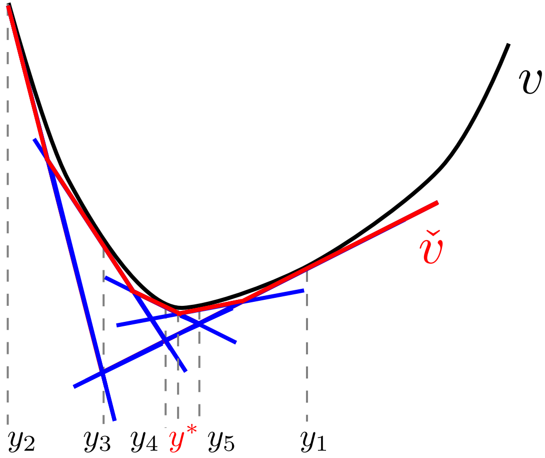

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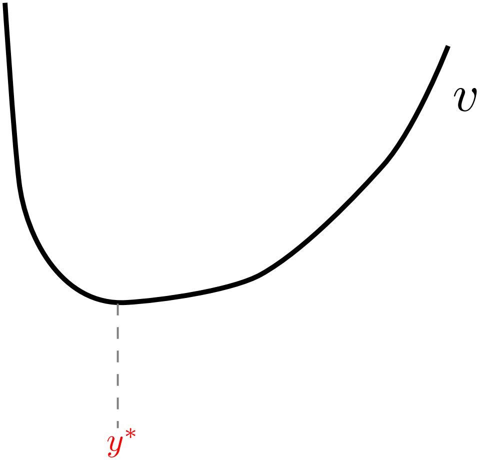

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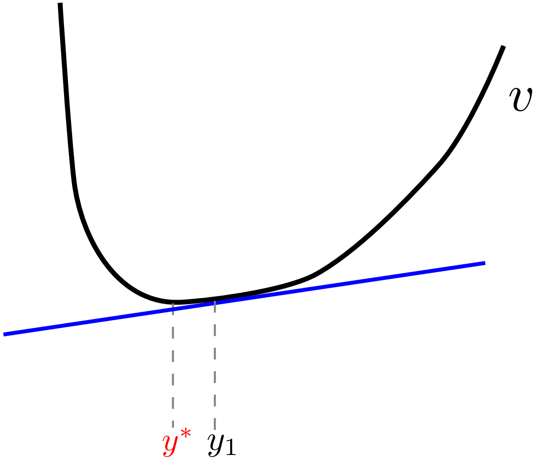

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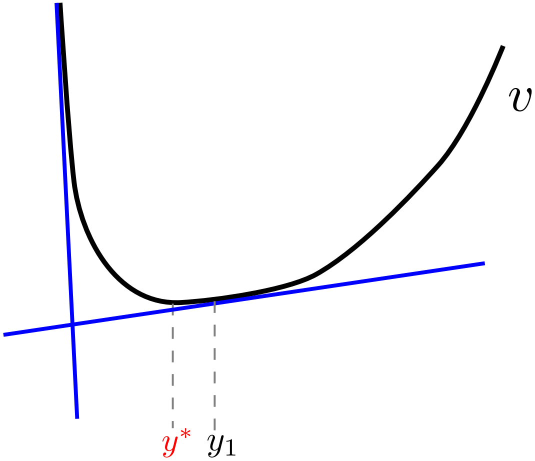

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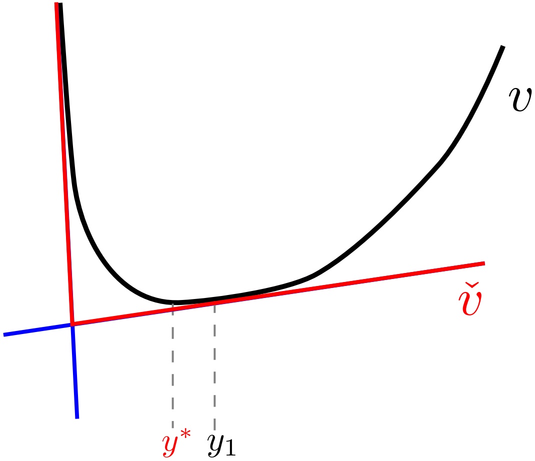

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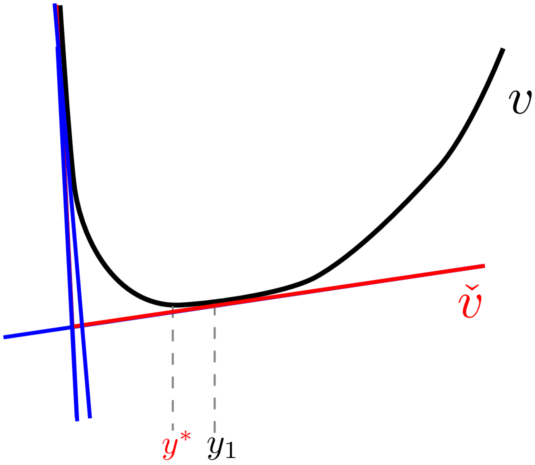

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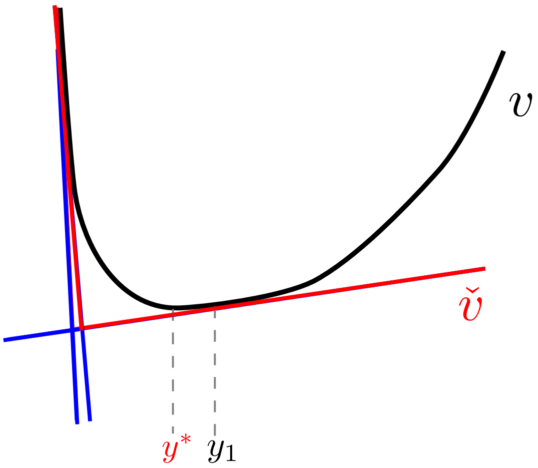

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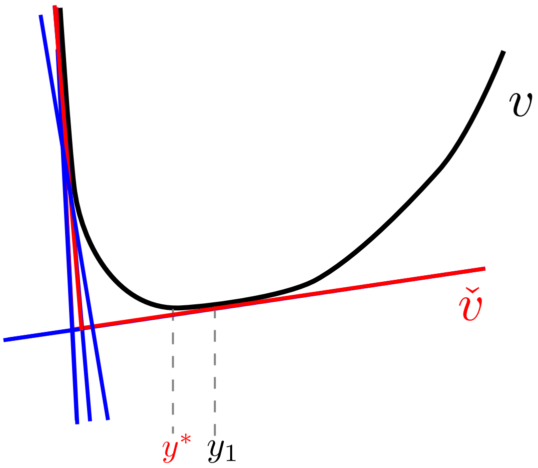

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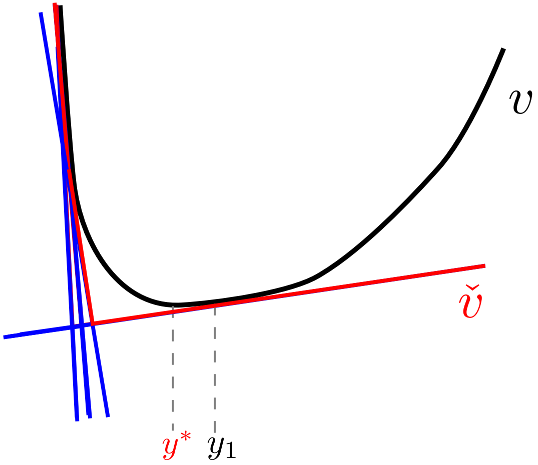

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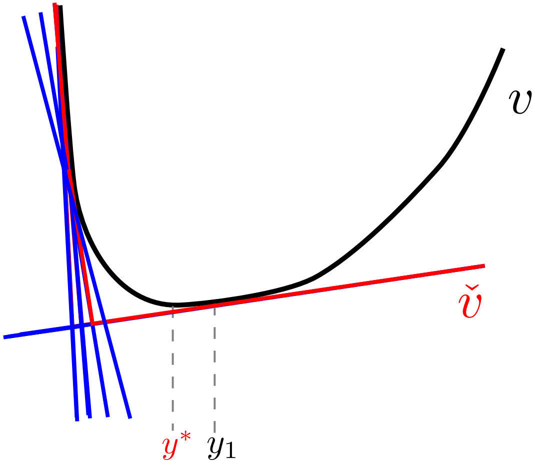

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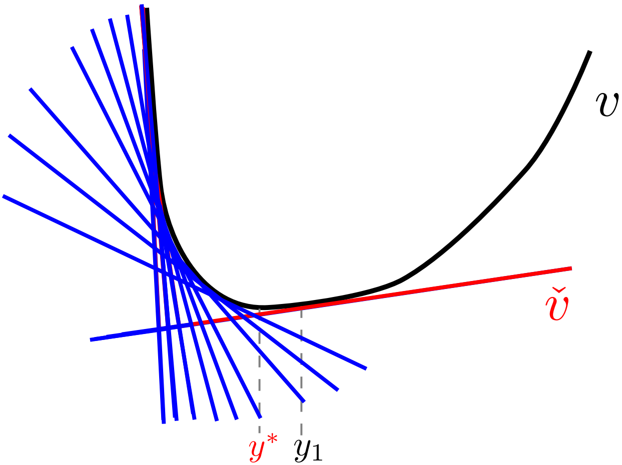


Outline

Stabilizations in the literature

The idea

Make the iterates stay close to the best one $\hat{y}_k$ (where we trust the model)

In nonsmooth convex optimization

- Equivalent of Benders method: Kelley method [Kelley 1960]
- Different stabilizations [Hiriart-Urruty and Lemaréchal 1993]

Proximal bundle	$y_{k+1} \in \arg\min_{y \in Y} \{ \check{v}_k(y) + 1/2t \, \|y - \hat{y}_k\|^2 \}$
Trust region	$y_{k+1} \in \arg\min_{y \in Y} \{ \check{v}_k(y) \text{ s.t. } \|y - \hat{y}_k\|^2 \leq R \}$
Level bundle	$y_{k+1} \in \arg\min_{y \in Y} \{ 1/2 \, \|y - \hat{y}_k\|^2 \text{ s.t. } \check{v}_k(y) \leq L \}$

In MILP, for Dantzig-Wolfe decomposition

Successful stabilization by proximal bundle method [Briant et al 2008]

Our proposal: level version of Benders method

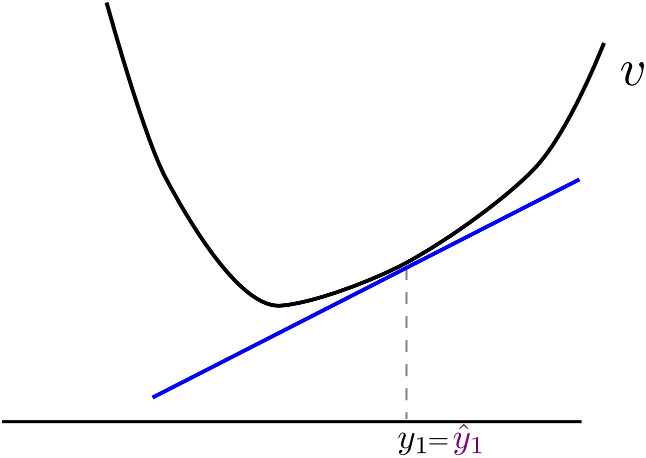
Level master subproblem

$$\begin{array}{ll} \min & \check{v}_k(y) \\ \text{s.t.} & y \in \check{V}_k \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \min & \frac{1}{2}\|y - \hat{y}_k\|_2^2 \\ \text{s.t.} & \check{v}_k(y) \leq \text{lev}_k \\ & y \in \check{V}_k \end{array}$$

What has changed?

- Integer Linear Program $\rightsquigarrow$ convex Integer Quadratic Program
- But: solvers are now good with convex IQPs
- Optimal value of the master no longer a lower bound
- But: when infeasible, *lev* is a lower bound of the optimal value

Illustration of level-Benders method



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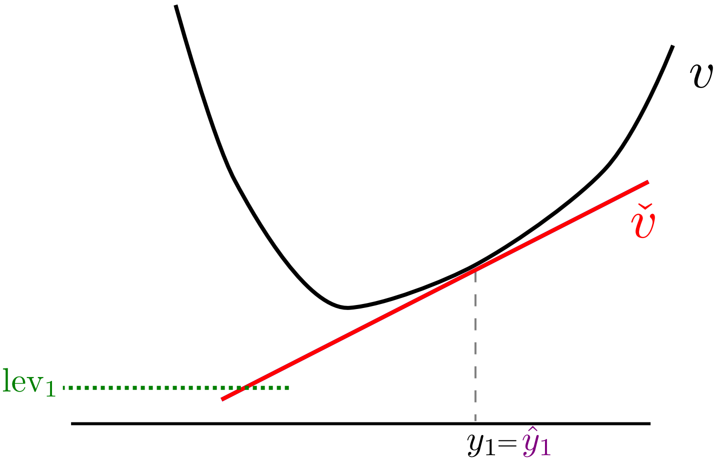

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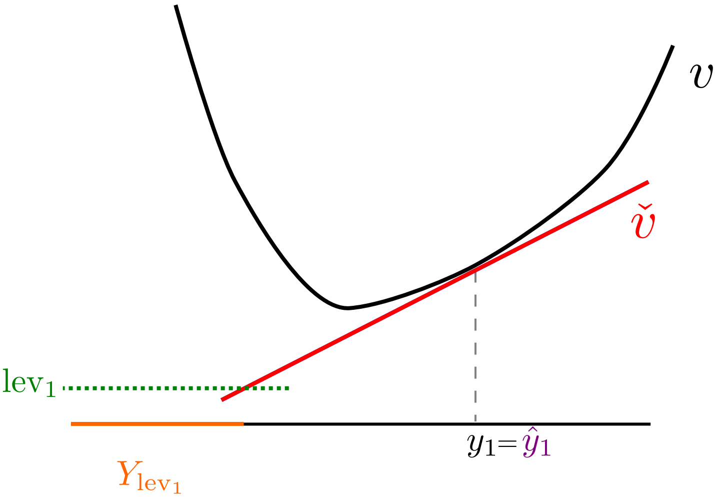

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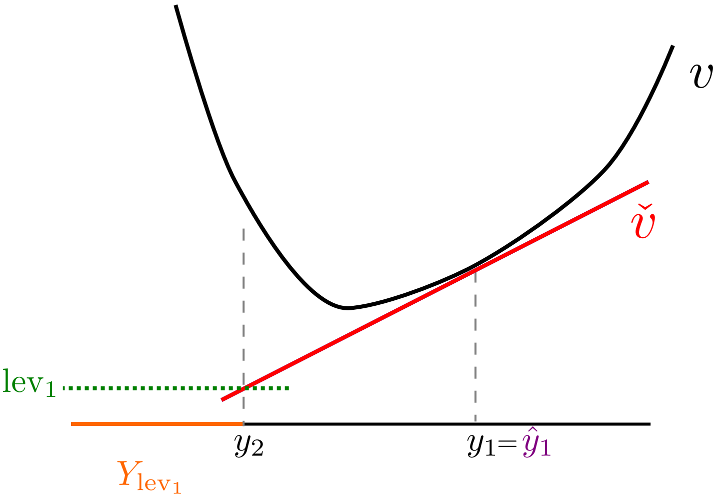

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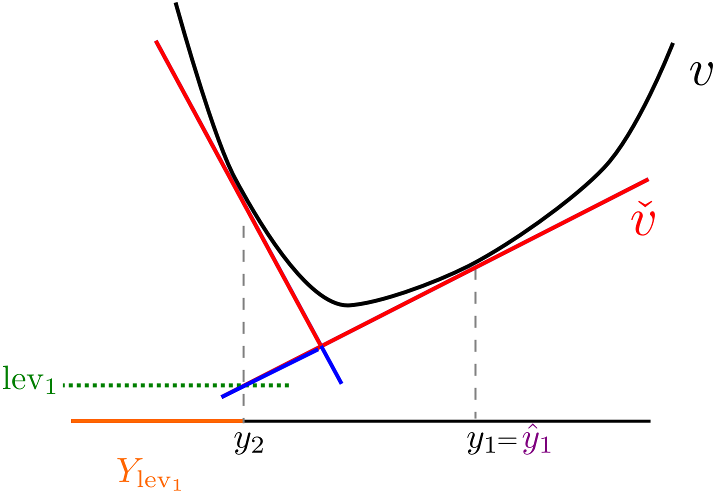

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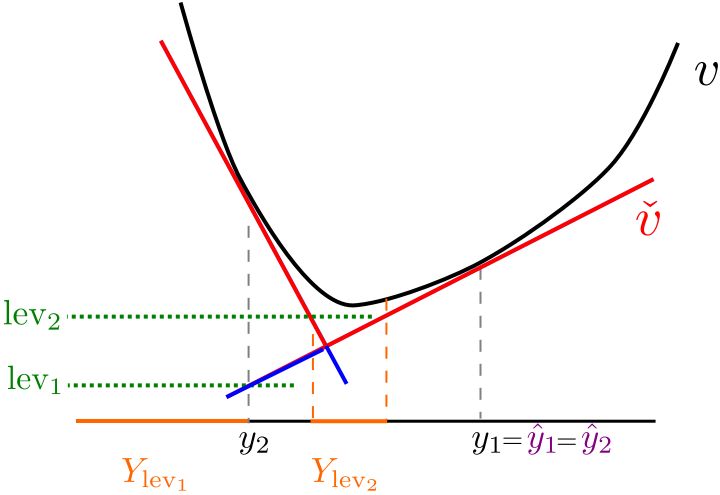

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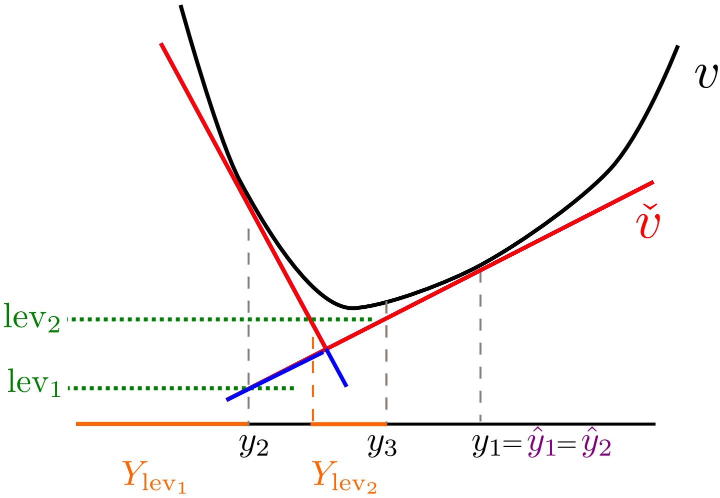

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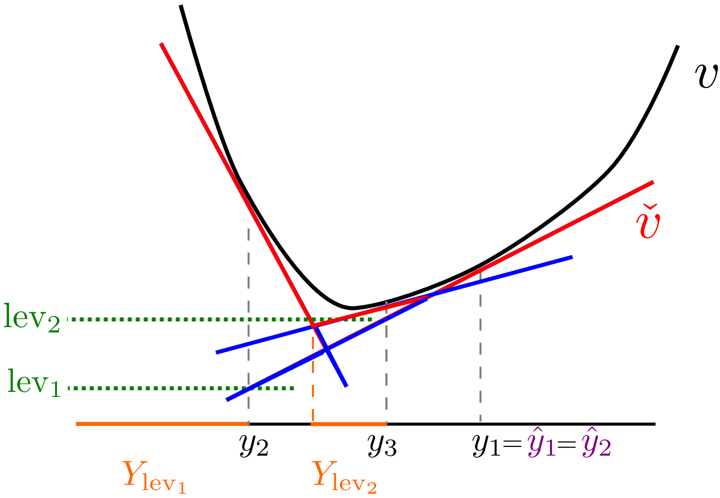

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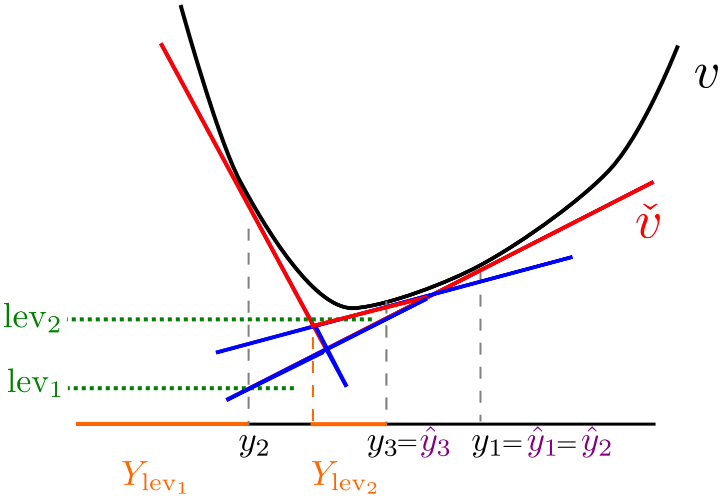

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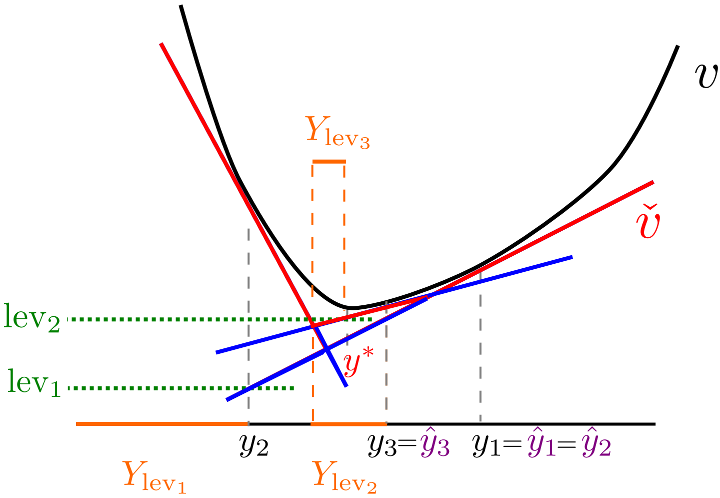
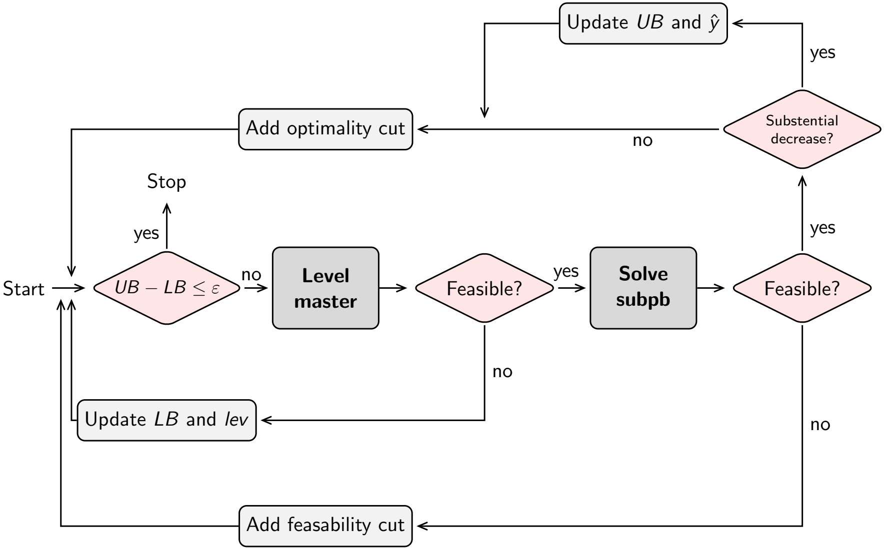


Level Benders algorithm



Convergence analysis

Convergence results are similar to GBD original results.

Theorem 1

If Y compact $\subset \text{dom}v$ then the algorithm asymptotically converges:

$$v(\hat{y}_k) \xrightarrow[k \to \infty]{} v^*$$

Theorem 2

If Y is finite then the algorithm converges in a finite number of iterations:

$$\exists k \;\text{ such that }\; v(\hat{y}_k) = v^*$$

Proof idea

Handle the 4 possible infinite loops:
infeasible master / infeasible subproblem / descent steps / null steps

Outline

Illustration on uncapacited network design problems

Instances

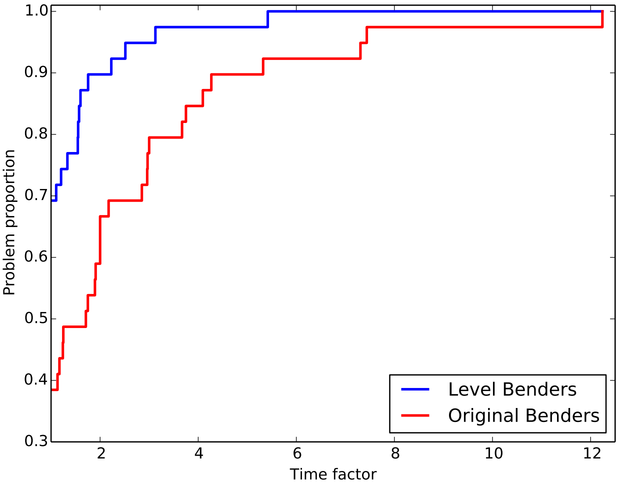
- 40 randomly generated instances
- Number of nodes: $N \leq 15$, number of commodities: $K \leq 15$

Typical results

Instances		Time (s)		Iterations	
N	K	Benders	LB	Benders	LB
5	3	0.11	0.17	14	25
8	5	1.24	1.37	36	33
10	5	8.15	3.75	75	43
10	8	216.20	40.61	218	86
15	5	88.28	11.87	135	46

Preliminary results

Performance profiles of Benders and level-Benders methods



Conclusion

Summary

- Level-benders method:
  - Exploits the MIQP great improvements
  - Same theoretical convergence as original Benders methods
  - Acceleration in practice
- Other nice practical features:
  - Inexact solving of subproblem
  - Limited memory

Compatible further accelerations

- Inexact (multi-)cuts
- Embedding the method in a branch and cut framework

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Thank you!