Position and Force Control of Nonsmooth Lagrangian Dynamical Systems without Friction

Sophie Chareyron and Pierre-Brice Wieber
INRIA Rhône-Alpes
38334 St-Ismier Cedex, France
{Sophie.Chareyron, Pierre-Brice.Wieber}@inria.fr

***Abstract*— Analyses of position and force control laws in the case of perfectly rigid bodies have been made so far with strong assumptions on the state of the contacts such as supposing that they are permanent. We're interested here in having a look at what happens when no such assumptions is made: we are led therefore to propose a Lyapunov stability analysis of a position and force control law in the mathematical framework of nonsmooth Lagrangian dynamical systems.**

I. Introduction

Many applications of robot manipulators require contact phases between the robots and their environments, and a regulation of both the position of the robots and the reaction forces at the contact points is usually demanded in this case. So far, analyses of the corresponding position and force control laws have been either focusing on robot manipulators and environments with finite stiffnesses [1] or they have been made in the case of perfectly rigid bodies with strong assumptions on the state of the contacts [2] such as supposing that they are permanent [3]. We're interested here in having a look at what happens in the case of perfectly rigid bodies when no such assumptions is made, and more precisely what happens with the propositions of [3].

Now, the development of a rigorous mathematical framework to study the dynamics of Lagrangian systems with perfectly rigid and non-permanent contacts is quite recent [4], [5], [6], [7] and uses mathematical tools which are still unusual in robot control theory. We are going therefore to spend some time in section II to present this mathematical framework building on convex and nonsmooth analysis. A very brief introduction to the general stability theory that can be proposed for such dynamical systems follows, and section III is entirely devoted to analysing in this nonsmooth dynamics framework the Lyapunov stability of the position and force control law proposed in [3].

II. Nonsmooth Lagrangian dynamical systems

A. Systems with non-permanent contacts

With n the number of degrees of freedom of the dynamical system, let's consider a time-variation of generalized coordinates $\boldsymbol{q} : \mathbb{R} \rightarrow \mathbb{R}^n$ and the related velocity $\dot{\boldsymbol{q}} : \mathbb{R} \rightarrow \mathbb{R}^n$:

$$\forall\, t, t_0 \in \mathbb{R},\ \boldsymbol{q}(t) = \boldsymbol{q}(t_0) + \int_{t_0}^{t} \dot{\boldsymbol{q}}(\tau)\, d\tau.$$

We're interested here with Lagrangian dynamical systems which may experience non-permanent contacts of perfectly rigid bodies. Geometrically speaking, the non-overlapping of rigid bodies can be expressed as a constraint on the position of the corresponding dynamical system, a constraint that will take the form here of a closed set $\Phi \subset \mathbb{R}^n$ in which the generalized coordinates are bound to stay [5]:

$$\forall\, t \in \mathbb{R},\ \boldsymbol{q}(t) \in \Phi.$$

This way, contact phases correspond to phases when $\boldsymbol{q}(t)$ lies on the boundary of Φ, and non-contact phases to phases when $\boldsymbol{q}(t)$ lies in the interior of Φ. We will suppose that this closed set is time-invariant, and we will have to suppose that it is convex for the stability analysis of section III.

We can define then for all $\boldsymbol{q} \in \Phi$ the tangent cone [8]

$$\mathcal{T}(\boldsymbol{q}) = \big\{ \boldsymbol{v} \in \mathbb{R}^n : \exists\, \tau_k \rightarrow 0,\ \tau_k > 0,\ \exists\, \boldsymbol{q}_k \rightarrow \boldsymbol{q},\ \boldsymbol{q}_k \in \Phi \ \text{with}\ \frac{\boldsymbol{q}_k - \boldsymbol{q}}{\tau_k} \rightarrow \boldsymbol{v} \big\},$$

and we can readily observe that if the velocity $\dot{\boldsymbol{q}}(t)$ has a left and right limit at an instant t, then obviously $-\dot{\boldsymbol{q}}^-(t) \in \mathcal{T}(\boldsymbol{q}(t))$ and $\dot{\boldsymbol{q}}^+(t) \in \mathcal{T}(\boldsymbol{q}(t))$.

Now, note that $\mathcal{T}(\boldsymbol{q}) = \mathbb{R}^n$ in the interior of the domain Φ, but it reduces to a half-space or even less on its boundary (Fig. 1): if the system reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$, it won't be able to continue its movement with a velocity $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$ and still stay in Φ (Fig. 1). A discontinuity of the velocity will have to occur then, corresponding to an impact between contacting rigid bodies, the landmark of *nonsmooth* dynamical systems.

We can also define for all $\boldsymbol{q} \in \Phi$ the normal cone [8]

$$\mathcal{N}(\boldsymbol{q}) = \big\{ \boldsymbol{v} \in \mathbb{R}^n :\ \forall\, \boldsymbol{q}' \in \Phi,\ \boldsymbol{v}^T(\boldsymbol{q}' - \boldsymbol{q}) \leq 0 \big\},$$

and we will see in the inclusion (4) of section II-C that it is directly related to the reaction forces arising from the contacts between rigid bodies.

Now, note that $\mathcal{N}(\boldsymbol{q}) = \{0\}$ in the interior of the domain Φ, and it contains at least a half-line of $\mathbb{R}^n$ on its boundary (Fig. 1): this will imply the obvious observation that non-zero contact forces may be experienced only on the boundary of the domain Φ, precisely when there is a contact. Discontinuities of the contact forces might be induced because of that, what will be discussed later.

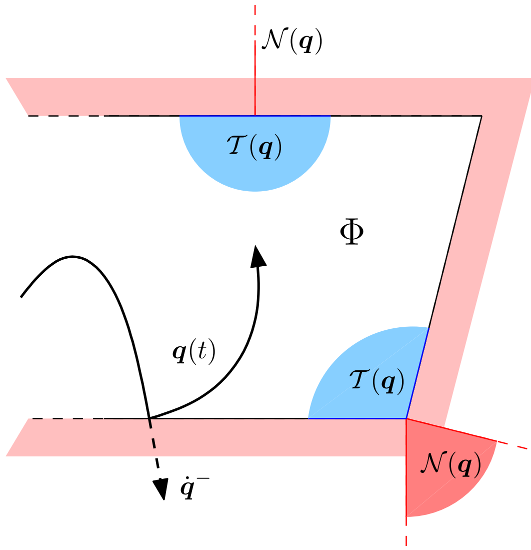

Fig. 1. Examples of tangent cones $\mathcal{T}(\boldsymbol{q})$ and normal cones $\mathcal{N}(\boldsymbol{q})$ on the boundary of the domain Φ, and example of a trajectory $\boldsymbol{q}(t) \in \Phi$ that reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$.

In the end, note that with these definitions, the state $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ appears now to stay inside the set

$$\Omega = \left\{(\boldsymbol{q}, \dot{\boldsymbol{q}}) : \ \boldsymbol{q} \in \Phi, \ \dot{\boldsymbol{q}} \in \mathcal{T}(\boldsymbol{q})\right\}.$$

B. Nonsmooth Lagrangian dynamics

The dynamics of Lagrangian systems subject to Lebesgues-integrable forces are usually expressed as differential equations,

$$\boldsymbol{M}(\boldsymbol{q}) \frac{d\dot{\boldsymbol{q}}}{dt} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \, \dot{\boldsymbol{q}} = \boldsymbol{f},$$

with $\boldsymbol{M}(\boldsymbol{q})$ the symmetric positive definite inertia matrix that we will suppose to be a C^1 function of $\boldsymbol{q}$, $\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \, \dot{\boldsymbol{q}}$ the corresponding nonlinear effects and $\boldsymbol{f}$ the Lebesgues-integrable forces. Classical solutions to these differential equations lead to smooth motions, with a locally absolutely continuous velocity $\dot{\boldsymbol{q}}(t)$.

But we have seen that discontinuities of the velocity may have to occur in the case of Lagrangian systems experiencing non-permanent contacts between rigid bodies. A mathematically rigorous way to allow such discontinuities in the dynamics of Lagrangian system has been proposed through measure differential equations [5], [9],

$$\boldsymbol{M}(\boldsymbol{q}) \, d\dot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \, \dot{\boldsymbol{q}} \, dt = \boldsymbol{f} \, dt + d\boldsymbol{r}, \tag{1}$$

with dt the Lebesgues measure and $d\boldsymbol{r}$ the reaction forces arising from the contacts between rigid bodies, an abstract measure which may not be Lebesgues-integrable. This way, the measure acceleration $d\dot{\boldsymbol{q}}$ may not be Lebesgues-integrable either so that the velocity may not be locally absolutely continuous anymore but only with locally bounded variations, $\dot{\boldsymbol{q}} \in \text{lbv}([t_0, T], \mathbb{R}^n)$ [5], [9] (for the sake of simplicity, solutions to these measure differential equations are considered only on compact time intervals).

Functions with locally bounded variations have left and right limits at every instant, and we have for every compact subinterval $[\sigma, \tau] \subset [t_0, T]$

$$\int_{[\sigma,\tau]} d\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\sigma).$$

Considering then the integral of the measure differential equations (1) over a singleton $\{\tau\}$, we have

$$\int_{\{\tau\}} \boldsymbol{M}(\boldsymbol{q}) \, d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q}) \int_{\{\tau\}} d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q}) \big(\dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\tau)\big),$$

$$\int_{\{\tau\}} \big(\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \, \dot{\boldsymbol{q}} - \boldsymbol{f}\big) dt = \big(\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \, \dot{\boldsymbol{q}} - \boldsymbol{f}\big) \int_{\{\tau\}} dt = 0,$$

leading to the following relationship between possible discontinuities of the velocities and possible atoms of the contact forces,

$$\boldsymbol{M}(\boldsymbol{q}) \left(\dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\tau)\right) = \int_{\{\tau\}} d\boldsymbol{r},$$

or, $\boldsymbol{M}(\boldsymbol{q})$ being invertible,

$$\dot{\boldsymbol{q}}^+(\tau) = \dot{\boldsymbol{q}}^-(\tau) + \boldsymbol{M}(\boldsymbol{q})^{-1} \int_{\{\tau\}} d\boldsymbol{r}. \tag{2}$$

C. Soft frictionless unilateral contacts

Following [5], we will consider that the non-permanent contacts that may be experienced by our Lagrangian systems are perfectly unilateral, frictionless and soft. Expressing the $\mathbb{R}^n$ valued measure $d\boldsymbol{r}$ as the product of a non-negative real measure $d\mu$ and a $\mathbb{R}^n$ valued function $\boldsymbol{r}'_\mu \in L^1_{loc}([t_0, T], d\mu; \mathbb{R}^n)$,

$$d\boldsymbol{r} = \boldsymbol{r}'_\mu \, d\mu, \tag{3}$$

the unilaterality of the contacts (no adhesive forces) together with the absence of friction corresponds to the inclusion

$$\forall \, t \in \mathbb{R}, \ -\boldsymbol{r}'_\mu(t) \in \mathcal{N}(\boldsymbol{q}(t)), \tag{4}$$

and the softness of the contacts, i.e. the fact that impacts are inelastic, corresponds to the complementarity condition

$$\forall \, t \in \mathbb{R}, \ \dot{\boldsymbol{q}}^+(t)^T \boldsymbol{r}'_\mu(t) = 0. \tag{5}$$

For a more in-depth presentation of these concepts and equations which are quite subtle, the interested reader should definitely refer to [5].

D. Some Lyapunov stability theory

The Lyapunov stability theory is usually presented for dynamical systems with states that vary continuously with time [10], [11]. Because of the possible discontinuities of their velocity, this might not be the case for the state $x(t) = (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ of nonsmooth Lagrangian dynamical systems. But the Lyapunov stability theory is in fact not strictly bound to continuity properties: using class $\mathcal{K}$ functions as defined in [10], we can state for example the following theorem that can be proved in a very similar way to what can be found in [10], [11],

Theorem : A closed invariant set $\mathcal{S} \subset \Omega$ is globally stable if and only if there exists a function $V : \Omega \to \mathbb{R}$ such that

(i) there exist two class $\mathcal{K}$ functions $\alpha(.)$ and $\beta(.)$ such that

$$\forall x \in \Omega,\ \alpha(d(x,\mathcal{S})) \leq V(x) \leq \beta(d(x,\mathcal{S})),$$

with $d(x,\mathcal{S})$ the distance between the state x and the set $\mathcal{S}$, and

(ii) for all solutions $x(t)$ to the nonsmooth dynamics (1), the function $V(x(t))$ is non-increasing with time.

Such a function is called a Lyapunov function with respect to the stable set $\mathcal{S}$.

Note now that the position and force control law that we are going to study in the next section is proved to be asymptotically stable in [3] through the use of LaSalle's invariance theorem. This latter is unfortunately tightly bound to the continuity of trajectories of the systems with respect to initial conditions, a property which doesn't hold for nonsmooth dynamical systems [7]. A theorem equivalent to LaSalle's for nonsmooth dynamical systems still doesn't exist, so we will stick here to the global stability proposed in the previous theorem.

III. Lyapunov stability analysis of a position and force control law

A. A position and force control law

Let's consider now that the Lebesgues-integrable forces $\boldsymbol{f}$ acting on the dynamics (1) consist of some external forces $\boldsymbol{e}_f$ and a control $\boldsymbol{u}$,

$$\boldsymbol{f} = \boldsymbol{e}_f + \boldsymbol{u}.$$

With the help of this control $\boldsymbol{u}$, we would like to stabilize both the position $\boldsymbol{q}$ of the dynamical systems and the reaction forces $d\boldsymbol{r}$ to some desired constant values $\boldsymbol{q}_d$ and $\boldsymbol{r}_d\,dt$ (following (3), the desired contact forces are defined through the product of the Lebesgues measure dt and a constant vector $\boldsymbol{r}_d \in \mathbb{R}^n$). First of all, these desired position and reaction forces have to be consistent with the contact model (4),

$$-\boldsymbol{r}_d \in \mathcal{N}(\boldsymbol{q}_d). \tag{6}$$

Following then the proposition of [3], we define the control $\boldsymbol{u}$ through the derivative of a strictly convex C^1 potential function $P(\boldsymbol{q})$, a dissipative term $\boldsymbol{C}\,\dot{\boldsymbol{q}}$ with $\boldsymbol{C}$ a positive definite matrix, and a compensation of the external forces,

$$\boldsymbol{u} = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}) - \boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{e}_f. \tag{7}$$

With this control law, the dynamics (1) becomes

$$\boldsymbol{M}(\boldsymbol{q})\,d\dot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\,dt = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\,dt - \boldsymbol{C}\,\dot{\boldsymbol{q}}\,dt + d\boldsymbol{r}, \tag{8}$$

the equilibria of which, with $\dot{\boldsymbol{q}} = 0$, are positions for which

$$0 = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\,dt + \boldsymbol{r}'_\mu\,d\mu.$$

This equation of measures is satisfied if and only if $d\mu = dt$ and

$$0 = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}) + \boldsymbol{r}'_\mu, \tag{9}$$

and through theorem VII.1.1.1 of [8], this corresponds together with (4) to the specification of the minima of $P(\boldsymbol{q})$ over the domain Φ: the equilibria of the closed loop dynamics correspond to the minima of the potential function. More precisely, since Φ is assumed to be convex and $P(\boldsymbol{q})$ strictly convex, if there is such a minimum then it is reached at a unique position: if there is an equilibrium position of the closed loop dynamics, then it is unique.

If we assume now that the potential function satisfies explicitely

$$\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}_d) = \boldsymbol{r}_d,$$

then there is such a mimimum through (6) and the same theorem of [8]: this minimum is $P(\boldsymbol{q}_d)$, reached at the position $\boldsymbol{q}_d$, and equation (9) becomes

$$0 = -\boldsymbol{r}_d + \boldsymbol{r}'_\mu,$$

so that the contact forces will be as desired at this equilibrium,

$$d\boldsymbol{r} = \boldsymbol{r}_d\,dt.$$

B. Lyapunov stability analysis

Since $P(\boldsymbol{q})$ has a global minimum reached at the unique position $\boldsymbol{q}_d$, with

$$K(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\,\dot{\boldsymbol{q}}$$

the kinetic energy of the dynamical system, the function

$$V(\boldsymbol{q},\dot{\boldsymbol{q}}) = K(\boldsymbol{q},\dot{\boldsymbol{q}}) + P(\boldsymbol{q}) - P(\boldsymbol{q}_d)$$

has 0 as a global minimum, reached at the unique state $(\boldsymbol{q}_d, 0)$.

Since it is convex with a minimum reached at a unique position, we know from proposition IV.3.2.5 and definition IV.3.2.6 of [8] that the function $P(\boldsymbol{q})$ is radially unbounded. Excluding pathological behaviours of the inertia matrix, we can suppose quite directly then that the function $V(\boldsymbol{q},\dot{\boldsymbol{q}})$ is also radially unbounded. Lemma 3.5 of [10] allows then to conclude that it satisfies condition (i) of the theorem of section II-D with respect to the set $\mathcal{S} = \{(\boldsymbol{q}_d, 0)\}$, appearing therefore as a possible Lyapunov function.

Indeed, classical differentiation rules of lbv functions [12], [5] allow to compute the time-derivative of the kinetic energy,

$$dK = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\,dt + \frac{\left(\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-\right)^T}{2}\boldsymbol{M}(\boldsymbol{q})\,d\dot{\boldsymbol{q}}.$$

For the closed loop dynamics (8), this time-derivative is

$$dK = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\big(\dot{\boldsymbol{M}}(\boldsymbol{q},\dot{\boldsymbol{q}}) - 2\,\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\big)\,\dot{\boldsymbol{q}}\,dt - \dot{\boldsymbol{q}}^T\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\,dt$$
$$- \dot{\boldsymbol{q}}^T\boldsymbol{C}\,\dot{\boldsymbol{q}}\,dt + \frac{\left(\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-\right)^T}{2}\,d\boldsymbol{r},$$

(note that $\dot{\boldsymbol{q}}^+dt = \dot{\boldsymbol{q}}^-dt = \dot{\boldsymbol{q}}\,dt$) where the first term is identically 0 since $\dot{\boldsymbol{M}}(\boldsymbol{q},\dot{\boldsymbol{q}}) - 2\,\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is an antisymmetric matrix and $-\dot{\boldsymbol{q}}^T\boldsymbol{C}\,\dot{\boldsymbol{q}}$ is non-positive since $\boldsymbol{C}$ is a positive matrix. Recalling then relations (2) and (3), we have

$$\frac{\left(\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-\right)^T}{2}\,d\boldsymbol{r} = \dot{\boldsymbol{q}}^{+T}d\boldsymbol{r} - \frac{1}{2}\left[\int_{\{\tau\}} d\boldsymbol{r}\right]^T\boldsymbol{M}(\boldsymbol{q})^{-1}d\boldsymbol{r}$$
$$= \dot{\boldsymbol{q}}^{+T}\boldsymbol{r}'_\mu\,d\mu$$
$$- \frac{1}{2}\left[\int_{\{\tau\}} d\mu\right]\boldsymbol{r}'_\mu{}^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{r}'_\mu\,d\mu$$

where the first term is identically 0 because of the complementarity condition (5) and the second term is non-positive since the inertia matrix is positive and $d\mu \geq 0$. All this ends up with

$$dK \leq -\dot{\boldsymbol{q}}^T\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\,dt,$$

and since the time-derivative of the potential function is precisely

$$dP = \dot{\boldsymbol{q}}^T\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\,dt,$$

we are led to

$$dV = dK + dP \leq 0.$$

The function $V(x(t))$ is therefore non-increasing with time, condition (ii) of the theorem of section II-D is also satisfied, and the proof that the state $(\boldsymbol{q}_d, 0)$ is globally stable with the closed loop dynamics (8) is completed.

Note that what we have proved here is the stability of the state $(\boldsymbol{q}_d, 0)$ only, and not of the contact forces $\boldsymbol{r}_d\,dt$: on the contrary to what appears in [3], non-zero contact forces can't be stable in our case since, as we have seen in section II-A, non-zero contact forces may be experienced only on the boundary of the domain Φ, when there is a contact. These forces may therefore jump to zero in every neighbourhood of any equilibrium position, what is not compatible with Lyapunov stability.

C. An example

Following [3], we can see for example that with a strictly convex quadratic potential function

$$P(\boldsymbol{q}) = \frac{1}{2}\,(\boldsymbol{q}-\boldsymbol{q}_d)^T\,\boldsymbol{W}\,(\boldsymbol{q}-\boldsymbol{q}_d) + \boldsymbol{r}_d^T(\boldsymbol{q}-\boldsymbol{q}_d)$$

with a symmetric positive definite matrix $\boldsymbol{W}$, the control law (7) becomes a strictly linear feedback

$$\boldsymbol{u} = -\boldsymbol{W}\,(\boldsymbol{q}-\boldsymbol{q}_d) - \boldsymbol{r}_d - \boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{e}_f$$

for which we know now that the equilibrium state $(\boldsymbol{q}_d, 0)$, where the contact forces are $\boldsymbol{r}_d\,dt$, is globally stable.

IV. Conclusion

We have seen that the position and force control law proposed in [3] can be prooved to be stable in the framework of nonsmooth dynamics with no need for any assumptions concerning the state of the contacts experienced by the systems.

This result is obtained with the help of differentiation rules for functions with locally bounded variations which are somehow different from the more usual ones for locally absolutely continuous functions, but which can be practiced in a very similar way, allowing to derive a Lyapunov stability analysis for nonsmooth dynamical systems very similar to what appears in the smooth case.

Extreme care must be taken though about the particularities of nonsmooth dynamical systems: if we can propose a stability theorem such as the one of section II-D, which is very similar to usual theorems for smooth dynamical systems, it doesn't mean that the whole stability theory for smooth dynamics can be translated to the nonsmooth case without specific and sometimes subtle adaptations. The example of Lasalle's theorem discussed in section II-D or the fact that the contact forces can't be stable for physical reasons speak for themselves.

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