Lecture 2. Time integration of Non Smooth Dynamical Systems (NSDS).

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May 31, 2006

Principle

Time-decomposition of the dynamics in

- *modes*, time-intervals in which the dynamics is smooth,
- discrete events, times where the dynamics is nonsmooth.

The following assumptions guarantee the existence and the consistency of such a decomposition

- The definition and the localization of the discrete events. The set of events is negligible with the respect to Lebesgue measure.
- The definition of time-intervals of non-zero lengths. the events are of finite number and "well-separated" in time. Problems with finite accumulations of impacts, or Zeno-state

Comments

On the numerical point of view, we need

- detect events with for instance root-finding procedure.
  - Dichotomy and interval arithmetic
  - Newton procedure for C^2 function and polynomials
- solve the non smooth dynamics at events with a reinitialization rule of the state,
- integrate the smooth dynamics between two events with any ODE solvers.

The smooth dynamics and the impact equations

The impact equations

The impact equations can be written at the time, t_i of discontinuities:

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{1}$$

This equation will be solved at the time of impact together with an impact law. That is for an Newton impact law

$$\begin{cases} M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\ U_{\mathrm{N}}^+(t_i) = \nabla_q h(q(t_i))v^+(t_i) \\ U_{\mathrm{N}}^-(t_i) = \nabla_q h(q(t_i))v^-(t_i) \\ p_i = \nabla_q^T h(q(t_i))P_{\mathrm{N},i} \\ 0 \leqslant U_{\mathrm{N}}^+(t_i) + eU_{\mathrm{N}}^-(t_i) \perp P_{\mathrm{N},i} \geqslant 0 \end{cases} \tag{2}$$

This problem can be reduced on the local unknowns $U_{\mathrm{N}}^+(t_i), P_{\mathrm{N},i}$ if the matrix $M(q(t_i))$ is assumed to be invertible. One obtains the following LCP at time t_i of discontinuities of v:

$$\begin{cases} U_{\mathrm{N}}^+(t_i) = \nabla_q h(q(t_i))(M(q(t_i)))^{-1}\nabla_q^T h(q(t_i))P_{\mathrm{N},i} + U_{\mathrm{N}}^-(t_i) \\ 0 \leqslant U_{\mathrm{N}}^+(t_i) + eU_{\mathrm{N}}^-(t_i) \perp P_{\mathrm{N},i} \geqslant 0 \end{cases} \tag{3}$$

The smooth dynamics and the impact equations

The smooth dynamics

The following smooth system are then to be solved $(dt - a.e.)$:

$$\begin{cases} M(q(t))\gamma^+(t) + F(t,q,v^+) = f^+(t) \\ g = g(q(t)) \\ f^+ = \nabla_q g(q(t))^T F^+(t) \\ 0 \leqslant g \perp F^+(t) \geqslant 0 \end{cases} \tag{1}$$

Reformulations of the unilateral constraints on Different kinematics levels

Differentiation of the constraints w.r.t time

The constraints $g = g(q(t))$ can de differentiate with respect to time as follows in the Lagrangian setting:

$$\begin{cases} \dot{g}^+ = U_{\mathrm{N}}^+ = \nabla_q g(q) v^+ \\ \ddot{g}^+ = \dot{U}_{\mathrm{N}}^+ = \Gamma_{\mathrm{N}} = \nabla_q g(q)\gamma^+ + \nabla_q \dot{g}(q) v^+ \end{cases} \tag{2}$$

Comments

Solving the smooth dynamics requires that the complementarity condition $0 \leqslant g \perp F^+(t) \geqslant 0$ must be written now at different kinematic level, i.e. in terms of right velocity U_{N}^+ and in terms of accelerations Γ_{N}^+.

Reformulations of the unilateral constraints on Different kinematics levels

At the velocity level

Assuming that U_{N}^{+} is right-continuous by definition of the right limit of a B.V. function, the complementarity condition implies, in terms of velocity, the following relation,

$$-F^{+} \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} > 0 \\ ]-\infty, 0] & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0 \end{cases} \quad . \tag{3}$$

A rigorous proof of this assertion can be found in GLOCKER (2001).

Equivalent formulations

- Inclusion into $N_{\mathbb{R}^+}(U_{\mathrm{N}}^+)$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ N_{\mathbb{R}^+}(U_{\mathrm{N}}^+) & \text{if } g = 0 \end{cases} \tag{3}$$

- Inclusion into $N_{T_{\mathbb{R}^+(g)}}(U_{\mathrm{N}}^+)$

$$-F^+ \in N_{T_{\mathbb{R}^+(g)}}(U_{\mathrm{N}}^+) \tag{4}$$

- In a complementarity formalism

$$\begin{array}{ll} \text{if } g = 0 & 0 \leqslant U_{\mathrm{N}}^+ \perp F^+ \geqslant 0 \\ \text{if } g > 0 & F^+ = 0 \end{array} \tag{5}$$

Reformulations of the unilateral constraints on Different kinematics levels

At the acceleration level

In the same way, the complementarity condition can be written at the acceleration level as follows.

$$-F^{+} \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0, \Gamma_{\mathrm{N}} > 0 \\ ]-\infty, 0] & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0, \Gamma_{\mathrm{N}} = 0 \end{cases} \tag{6}$$

A rigorous proof of this assertion can be found in Glocker (2001).

Reformulations of the unilateral constraints on Different kinematics levels

Equivalent formulations

- Inclusion into a cone $N_{\mathbb{R}^+}(\Gamma_{\mathrm{N}})$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^+ > 0 \\ N_{\mathbb{R}^+}(\Gamma_{\mathrm{N}}) \end{cases} \tag{6}$$

- Inclusion into $N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n)$

$$-F^+ \in N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n) \tag{7}$$

- In the complementarity formalism,

$$\begin{array}{ll} \text{if } g = 0, U_{\mathrm{N}}^+ = 0 & 0 \leqslant \Gamma_{\mathrm{N}}^+ \perp F^+ \geqslant 0 \\ \text{otherwise} & F^+ = 0 \end{array} \tag{8}$$

Reformulations of the smooth dynamics at acceleration level.

The smooth dynamics as an inclusion

$$\begin{cases} M(q(t))\gamma^+(t) + F(t,q,v^+) = f^+(t) \\ \Gamma_{\mathrm{N}} = \nabla_q g(q)\gamma^+ + \nabla_q \dot{g}(q) v^+ \\ f^+(t) = \nabla_q g(q(t))^T F^+(t) \\ -F^+ \in N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n) \end{cases} \tag{9}$$

Reformulations of the smooth dynamics at acceleration level.

The smooth dynamics as a LCP

When the condition, $g = 0$, $U_{\mathrm{N}}^{+} = 0$ is satisfied, we obtain the following LCP

$$\begin{cases} M(q(t))\gamma^{+}(t) + F(t,q,v^{+}) = \nabla_q g(q(t))^T F^{+}(t) \\ \\ \Gamma_{\mathrm{N}}^{+} = \nabla_q g(q)\gamma^{+} + \nabla_q \dot{g}(q) v^{+} \\ \\ 0 \leqslant \Gamma_{\mathrm{N}}^{+} \perp F^{+} \geqslant 0 \end{cases} \tag{10}$$

which can be reduced on variable Γ_{N}^{+} and F^{+}, if $M(q(t))$ is invertible,

$$\begin{cases} \Gamma_{\mathrm{N}}^{+} = \nabla_q g(q) M^{-1}(q(t))(-F(t,q,v^{+})) + \nabla_q \dot{g}(q) v^{+} \\ \qquad\qquad + \nabla_q g(q) M^{-1} \nabla_q g(q(t))^T F^{+}(t) \\ \\ 0 \leqslant \Gamma_{\mathrm{N}}^{+} \perp F^{+} \geqslant 0 \end{cases} \tag{11}$$

The case of a single contact.

Two modes for the non smooth dynamics

1. *The constraint is not active.* $F^+ = 0$

$$M(q)\gamma^+ + F(\cdot, q, v) = 0 \tag{12}$$

   In this case, we associate to this step an integer, $status_k = 0$.

2. *The constraint is active.* Bilateral constraint $\Gamma_N^+ = 0$,

$$\begin{bmatrix} M(q) & -\nabla_q g(q)^T \\ \nabla_q g(q) & 0 \end{bmatrix} \begin{bmatrix} \gamma^+ \\ F^+ \end{bmatrix} = \begin{bmatrix} -F(\cdot, q, v) \\ \nabla_q \dot{g}(q) v^+ \end{bmatrix} \tag{13}$$

   In this case, we associate to this step an integer, $status_k = 1$.

The case of a single contact.

[Case 1] $status_k = 0$.

Integrate the system (12) on the time interval $[t_k, t_{k+1}]$

Case 1.1 $g_{k+1} > 0$
The constraint is still not active. We set $status_{k+1} = 0$.

Case 1.2 $g_{k+1} = 0$, $U_{N,k+1} < 0$
In this case an impact occurs. The value $U_{N,k+1} < 0$ is considered as the pre-impact velocity U^- and the impact equation (3) is solved. After, we set $U_{N,k+1} = U^+$. Two cases are then possible:

- Case 1.2.1 $U_+ > 0$
  Just after the impact, the relative velocity is positive. The constraint ceases to be active and we set $status_{k+1} = 0$.
- Case 1.2.2 $U_+ = 0$
  The relative post-impact velocity vanishes. In the case, in order to determine the new status, we solve the LCP (10) to obtain. three cases are then possible:
  - Case 1.2.2.1 $\Gamma_{N,k+1} > 0$, $F_{k+1} = 0$
    The constraint is still not active. We set $status_{k+1} = 0$.
  - Case 1.2.2.2 $\Gamma_{N,k+1} = 0$, $F_{k+1} > 0$
    The constraint has to be activated. We set $status_{k+1} = 1$.
  - Case 1.2.2.3 $\Gamma_{N,k+1} = 0$, $F_{k+1} = 0$
    This case is undetermined. We need to know the value of $\dot{\Gamma}_N^+$.

The case of a single contact.

[Case 1] $status_k = 0$.

Integrate the system (12) on the time interval $[t_k, t_{k+1}]$

Case 1.3 $g_{k+1} = 0,\ U_{\mathrm{N},k+1} = 0$

In this case, we have a grazing constraint. To known what should be the status for the future time, we compute the value of $\Gamma_{\mathrm{N},k+1}, F_{k+1}$ thanks to the LCP (10) assuming that $U^+ = U^- = U_{\mathrm{N},k+1}$. Three cases are then possible:

- **Case 1.3.1** $\Gamma_{\mathrm{N},k+1} > 0,\ F_{k+1} = 0$
  The constraint is still not active. We set $status_{k+1} = 0$.
- **Case 1.3.2** $\Gamma_{\mathrm{N},k+1} = 0,\ F_{k+1} > 0$
  The constraint has to be activated. We set $status_{k+1} = 1$.
- **Case 1.3.3** $\Gamma_{\mathrm{N},k+1} = 0,\ F_{k+1} = 0$
  This case is undetermined. We need to know the value of $\dot{\Gamma}^+_{\mathrm{N}}$.

Case 1.4 $g_{k+1} = 0,\ U_{\mathrm{N},k+1} < 0$

The activation of the constraint has not been detected. We seek for the first time t_* such that $g = 0$. We set $t_{k+1} = t_*$. Then we perform all of these procedure keeping $status_k = 0$.

Case 1.5 $g_{k+1} < 0$

The activation of the constraint has not been detected. We seek for the first time t_* such that $g = 0$. We set $t_{k+1} = t_*$. Then we perform all of these procedure keeping $status_k = 0$.

[Case 2] $status_k = 1$

Integrate the system (13) on the time interval $[t_k, t_{k+1}]$

Case 2.1 $g_{k+1} \neq 0$ or $U_{\mathrm{N},k+1} = 0$
Something is wrong in the time integration or the drift from the constraints is too huge.

Case 2.2 $g_{k+1} = 0$, $U_{\mathrm{N},k+1} = 0$
In this case, we assume that $U^+ = U^- = U_{\mathrm{N},k+1}$ and we compute $\Gamma_{\mathrm{N},k+1}, F_{k+1}$ thanks to the LCP (10) assuming that $U^+ = U^- = U_{\mathrm{N},k+1}$. Three cases are then possible

- **Case 2.2.1** $\Gamma_{\mathrm{N},k+1} = 0$, $F_{k+1} > 0$
  The constraint is still active. We set $status_{k+1} = 1$.
- **Case 2.2.2** $\Gamma_{\mathrm{N},k+1} > 0$, $F_{k+1} = 0$
  The bilateral constraint is no longer valid. We seek for the time t_* such that $F^+ = 0$. We set $t_{k+1} = t_*$ and we perform the integration up to this instant. We perform all of these procedure at this new time t_{k+1}
- **Case 2.2.3** $\Gamma_{\mathrm{N},k+1} = 0$, $F_{k+1} = 0$
  This case is undetermined. We need to know the value of $\dot{\Gamma}_{\mathrm{N}}^+$.

The case of a single contact.

Comments

- The Delassus example.
  In the one-contact case, a naive approach consists in to suppressing the constraint $F_{k+1} = 0 < 0$ after a integration with a bilateral constraints.
  ➜ Work only for the one contact case.
- The role of the "ε"
  In practical situation, all of the test are made up to an accuracy threshold. All statements of the type $g = 0$ are replaced by $|g| < \varepsilon$. The role of these epsilons can be very important and they are quite difficult to size.

The case of a single contact.

Comments

- If the ODE solvers is able to perform the root finding of the function $g = 0$ for $status_k = 0$ and $F^+ = 0$ for $status_k = 1$
  ➜ the case 1.4, 1.5 and the case 2.2.2 can be suppressed in the decision tree.
- If the drift from the constraints is also controlled into the ODE solver by a error computation,
  ➜ the case 2.1 can also be suppressed
- Most of the case can be resumed into the following step
  - Continue with the same status
  - Compute $U_{\mathrm{N},k+1}, P_{k+1}$ thanks to the LCP (3)(impact equations).
  - Compute $\Gamma_{\mathrm{N},k+1}, F_{k+1}$ thanks to the LCP (10) (Smooth dynamics)

➜ Rearranging the cases, we obtain the following algorithm.

The case of a single contact. An algorithm

Require: $(g_k, U_{\mathrm{N},k}, status_k)$
Ensure: $(g_{k+1}, U_{\mathrm{N},k+1}, status_{k+1})$

Time-integration of the system on $[t_k, t_{k+1}]$(12) if $status_k = 0$ or of the system (13) if $status_k = 1$ up to an event.

if $g_{k+1} > 0$ **then**
  $status_{k+1} = 0$ *//The constraint is still not active. (case 1.1)*
end if
if $g_{k+1} = 0, U_{\mathrm{N},k+1} < 0$ **then**
  //The constraint is active $g_{k+1} = 0$ and an impact occur $U_{\mathrm{N},k+1} < 0$ (case 1.2)
  Solve the LCP (3) for $U_{\mathrm{N}}^- = U_{\mathrm{N},k+1}$; $U_{\mathrm{N},k+1} = U_{\mathrm{N}}^+$
  if $U_{\mathrm{N},k+1} > 0$ **then** $status_{k+1} = 0$
end if
if $g_{k+1} = 0, U_{\mathrm{N},k+1} = 0$ **then**
  //The constraint is active $g_{k+1} = 0$ without impact (case 1.2.2, case 1.3, case 2.2)
  solve the LCP (11)
  if $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} > 0$ **then**
    $status_{k+1} = 1$
  else if $\Gamma_{\mathrm{N},k+1} > 0, F_{k+1} = 0$ **then**
    $status_{k+1} = 0$
  else if $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} = 0$ **then**
    //Undetermined case.
  end if
end if
Go to the next time step

The multi-contact case and the index-sets

Index sets

The index set I is the set of all unilateral constraints in the system

$$I = \{1 \dots \nu\} \subset \mathbb{N} \tag{14}$$

The index-set I_c is the set of all active constraints of the system,

$$I_c = \{\alpha \in I, g^\alpha = 0\} \subset I \tag{15}$$

and the index-set I_s is the set of all active constraints of the system with a relative velocity equal to zero,

$$I_s = \{\alpha \in I_c, U_{\mathrm{N}}^\alpha = 0\} \subset I_c \tag{16}$$

The multi-contact case and the index-sets

Impact equations

$$
\begin{cases}
M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\
U_{\mathrm{N}}^+(t_i) = \nabla_q g(q(t_i))v^+(t_i) \\
U_{\mathrm{N}}^-(t_i) = \nabla_q g(q(t_i))v^-(t_i) \\
p_i = \nabla_q^T g(q(t_i))P_{\mathrm{N},i} \\
\\
P_{\mathrm{N},i}^\alpha = 0; U_{\mathrm{N}}^{\alpha,+}(t_i) = U_{\mathrm{N}}^{\alpha,-}(t_i), \quad \forall \alpha \in I \setminus I_c \\
\\
0 \leqslant U_{\mathrm{N}}^{+,\alpha}(t_i) + eU_{\mathrm{N}}^{-,\alpha}(t_i) \perp P_{\mathrm{N},i}^\alpha \geqslant 0, \quad \forall \alpha \in I_c
\end{cases}
\tag{17}
$$

Using the fact that $P_{\mathrm{N},i}^\alpha = 0$ for $\alpha \in I \setminus I_c$, this problem can be reduced on the local unknowns $U_{\mathrm{N}}^+(t_i), P_{\mathrm{N},i} \quad \forall \alpha \in I_c$.

The multi-contact case and the index-sets

Modes for the smooth Dynamics

- The smooth unilateral dynamics as a LCP

$$\begin{cases} M(q)\gamma^+ + F_{int}(\cdot, q, v) = F_{ext} + \nabla_q g(q)^T F^+ \\ \Gamma_N^+ = \nabla_q g(q)\gamma^+ + \nabla_q \dot{g}(q) v^+ \\ F^{+,\alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ 0 \leqslant \Gamma_N^{+,\alpha} \perp F^{+,\alpha} \geqslant 0 \quad \forall \alpha \in I_s \end{cases} \tag{18}$$

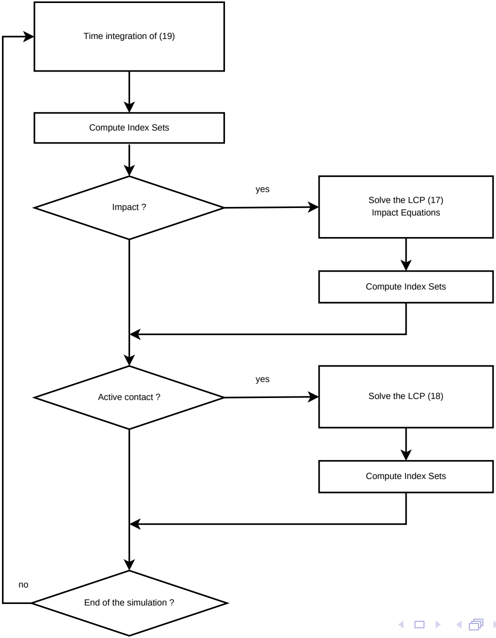
- The smooth bilateral dynamics

$$\begin{cases} M(q)\gamma^+ + F_{int}(\cdot, q, v) = F_{ext} + \nabla_q g(q)^T F^+ \\ \Gamma_N^+ = \nabla_q g(q)\gamma^+ + \nabla_q \dot{g}(q) v^+ \\ F^{+,\alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ \Gamma_N^{+,\alpha} = 0 \quad \forall \alpha \in I_s \end{cases} \tag{19}$$

Require: $(g_k, U_{\mathrm{N},k}, I_{c,k}, I_{s,k})$,
Ensure: $(g_{k+1}, U_{\mathrm{N},k+1}, I_{c,k+1}, I_{s,k+1})$

Time-integration on $[t_k, t_{k+1}]$ of the system (19) according to $I_{c,k}$ and $I_{s,k}$ up to an event.
Compute the temporary index-sets $I_{c,k+1}$ and $I_{s,k+1}$.
if $I_{c,k+1} \setminus I_{s,k+1} \neq \emptyset$ **then**
  //Impacts occur.
  Solve the LCP (17).
  Update the index-set $I_{c,k+1}$ and temporary $I_{s,k+1}$
  Check that $I_{c,k+1} \setminus I_{s,k+1} = \emptyset$
end if
if $I_{s,k+1} \neq \emptyset$ **then**
  Solve the LCP (18)
  for $\alpha \in I_{s,k+1}$ **do**
    if $\Gamma_{\mathrm{N},\alpha,k+1} > 0, F_{\alpha,k+1} = 0$ **then**
      remove α from $I_{s,k+1}$ and $I_{c,k+1}$
    else if $\Gamma_{\mathrm{N},\alpha,k+1} = 0, F_{\alpha,k+1} = 0$ **then**
      //Undetermined case.
    end if
  end for
end if
// Go to the next time step

The multi-contact case and the index-sets



Comments and extensions

Extensions to Coulomb's friction

The set I_r is the set of sticking or rolling contact:

$$I_r = \{\alpha \in I_s, U_{\rm N}^{\alpha} = 0, \|U_{\rm T}\| = 0\} \subset I_s, \quad (20)$$

is the set of sticking or rolling contact, and

$$I_t = \{\alpha \in I_s, U_{\rm N}^{\alpha} = 0, \|U_{\rm T}\| > 0\} \subset I_s, \quad (21)$$

is the set of slipping or sliding contact.

Remarks

In the 3D case, checking the events and the transition sticking/sliding and sliding/sticking is not a easy task.

Comments

Advantages and Weaknesses and the Event Driven schemes

- Advantages :
  - Low cost implementation of time integration solvers (re-use of existing ODE solvers).
  - Higher-order accuracy on free motion.
  - Pseudo-localization of the time of events with finite time-step.
- Weaknesses
  - Numerous events in short time.
  - Accumulation of impacts.
  - No convergence proof
  - Robustness with the respect to thresholds "ε". Tuning codes is difficult.

Principle of Time–stepping schemes

1. A unique formulation of the dynamics is considered. For instance, for the Lagrangian systems, a dynamics in terms of measures.

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = dr \\ v^+ = \dot{q}^+ \end{cases} \tag{22}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k,t_{k+1}]} dv = \int_{]t_k,t_{k+1}]} dv = (v^+(t_{k+1}) - v^+(t_k)) \approx (v_{k+1} - v_k) \tag{23}$$

3. Consistent approximation of measure inclusion.

$$-dr \in N_{T_C(q(t))}(v^+(t)) \tag{24}$$

$$\rightarrow \begin{cases} p_{k+1} \approx \int_{]t_k,t_{k+1}]} dr \\ p_{k+1} \in N_{T_C(q_k)}(v_{k+1}) \end{cases} \tag{25}$$

The Moreau's catching–up algorithm for the first order sweeping process

Catching–up algorithm

Let us consider the first order sweeping process with a B.V. solution:

$$\begin{cases} -du \in N_{K(t)}(u(t)) \quad (t \geqslant 0), \\ u(0) = u_0. \end{cases} \tag{26}$$

The so-called "Catching–up algorithm" is defined in Moreau (1977):

$$-(u_{k+1} - u_k) \in \partial\psi_{K(t_{k+1})}(u_{k+1}) \tag{27}$$

where u_k stands for the approximation of the right limit of u at t_k.
By elementary convex analysis, this is equivalent to:

$$u_{k+1} = prox(K(t_{k+1}), u_k). \tag{28}$$

The Moreau's catching–up algorithm for the first order sweeping process

Difference with an backward Euler scheme

- the catching–up algorithm is based on the evaluation of the measure du on the interval $]t_k, t_{k+1}]$, i.e. $du(]t_k, t_{k+1}]) = u^+(t_{k+1}) - u^+(t_k)$.
- the backward Euler scheme is based on the approximation of $\dot{u}(t)$ which is not defined in a classical sense for our case.

When the time step vanishes, the approximation of the measure du tends to a finite value corresponding to the jump of u. Particularly, this fact ensures that we handle only finite values.

Higher order approximation

Higher order schemes are meant to approximate the n-th derivative of the discretized function. Non sense for a non smooth solution.

Mathematical results

For Lipschitz and RCBV sweeping processes, convergence and consistency results are based on the catching–up algorithm.
Monteiro Marques (1993) ; Kunze & Monteiro Marqus (2000)

The Moreau's catching–up algorithm for the first order sweeping process

Time-independent convex set K

Let us recall now the UDI

$$-(\dot{x}(t) + f(x(t)) + g(t)) \in \mathbb{N}_K(x(t)), \quad x(0) = x_0 \tag{29}$$

In the same way, the inclusion can be discretized by

$$-(x_{k+1} - x_k) + h(f(x_{k+1}) + g(t_{k+1})) = \mu_{k+1} \in \mathbb{N}_K(x_{k+1}), \tag{30}$$

- In this discretization, an evaluation of the measure dx by the approximates value μ_{k+1}.
- If the initial condition does not satisfy the inclusion at the initial time, the jump in the state can be treated in a consistent way.

The Moreau's catching–up algorithm for the first order sweeping process

Time-independent convex set $K = \mathbb{R}^n_+$

The previous problem can be written as a special non linear complementarity problem:

$$\begin{cases} (x_{k+1} - x_k) - h(f(x_{k+1}) + g(t_{k+1})) = \mu_{k+1} \\ 0 \leqslant x_{k+1} \perp \mu_{k+1} \geqslant 0 \end{cases} \quad (31)$$

If $f(x) = Ax$ we obtain the following LCP(q,M):

$$\begin{cases} (I - hA)x_{k+1} - (x_k + hg(t_{k+1})) = \mu_{k+1} \\ 0 \leqslant x_{k+1} \perp \mu_{k+1} \geqslant 0 \end{cases} \quad (32)$$

with $M = (I - hA)$ and $q = -(x_k + hg(t_{k+1}))$.

Remark

It is noteworthy that the value μ_{k+1} approximates the measure $d\lambda$ on the time interval rather than directly the value of λ.

The Moreau's catching–up algorithm for the first order sweeping process

Remark

Particularly, if the set K is polyhedral by :

$$K = \{x, Cx \geqslant 0\} \tag{33}$$

If a constraint qualification holds, the DI (29) in the linear case $f(x) = -Ax$ is equivalent the the following LCS:

$$\begin{cases} \dot{x} = Ax + C^T\lambda \\ y = Cx \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{34}$$

In this case, the catching–up algorithms yields:

$$\begin{cases} x_{k+1} - x_k = hAx_{k+1} + C^T\mu^{k+1} \\ y_{k+1} = Cx_{k+1} \\ 0 \leqslant y_{k+1} \perp \mu_{k+1} \geqslant 0 \end{cases} \tag{35}$$

We will see later in Section 3 that this discretization is very similar to the discretization proposed by Camlibel *et al.* (2002) for LCS.

Time stepping scheme for Linear Complementarity Systems (LCS)

Backward Euler scheme

Starting from the LCS

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{36}$$

Camlibel *et al.* (2002) apply a backward Euler scheme to evaluate the time derivative $\dot{x}$ leading to the following scheme:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{h} = Ax_{k+1} + B\lambda_{k+1} \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1} \\ 0 \leqslant \lambda_{k+1} \perp y_{k+1} \geqslant 0 \end{cases} \tag{37}$$

which can be reduced to a LCP by a straightforward substitution:

$$0 \leqslant \lambda_{k+1} \perp C(I - hA)^{-1}x_k + (hC(I - hA)^{-1}B + D)\lambda_{k+1} \geqslant 0 \tag{38}$$

Time stepping scheme for Linear Complementarity Systems (LCS)

Convergence results

If D is nonnegative definite or that the triplet (A, B, C) is observable and controllable and (A, B, C, D) is positive real, they exhibit that some subsequences of $\{y_k\}, \{\lambda_k\}, \{x_k\}$ converge weakly to a solution y, λ, x of the LCS. CAMLIBEL *et al.* (2002)

Such assumptions imply that the relative degree r is less or equal to 1.

Remarks

- In the case of the relative degree 0, the LCS is equivalent to a standard system of ODE with a Lipschitz-continuous r.h.s field. The result of convergence is then similar to the standard result of convergence for the Euler backward scheme.
- In the case of a relative degree equal to 1, the initial condition must satisfy the unilateral constraints $y_0 = Cx_0 \geqslant 0$. Otherwise, the approximation $\dfrac{x_{k+1} - x_k}{h}$ has non chance to converge if the state possesses a jump. This situation is precluded in the result of convergence in (CAMLIBEL *et al.*, 2002).

Time stepping scheme for Linear Complementarity Systems (LCS)

Remark

Following the remark 5, we can note some similarities with the catching–up algorithm. Two main differences have however to be noted:

- the first one is that the sweeping process can be equivalent to a LCS under the condition $C = B^T$. In this way, the previous time-stepping scheme extend the catching–up algorithm to more general systems.
- The second major discrepancy is a s follows. The catching–up algorithm does not approximate directly the time-derivative $\dot{x}$ as

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h} \tag{39}$$

  but directly the measure of the time interval by

$$dx(]t, t+h]) = x^+(t+h) - x^+(t) \tag{40}$$

  This difference leads to a consistent time-stepping scheme if the state possesses an initial jump. A direct consequence is that the primary variable μ_{k+1} in the catching up algorithm is homogeneous to a measure of the time-interval.

Time stepping scheme for Linear Complementarity Systems (LCS)

θ–method

In the case of a relative degree 0, the following scheme based on a $\theta-$method ($\theta \in [0,1]$) should work also

$$\begin{cases} \dfrac{x_{k+1} - x_k}{h} = A(\theta x_{k+1} + (1-\theta)x_k) + B(\theta\lambda_{k+1} + (1-\theta)\lambda_k) \\[2ex] y_{k+1} = Cx_{k+1} + D\lambda_{k+1} \\[2ex] 0 \leqslant \lambda_{k+1} \perp w_{k+1} \geqslant 0 \end{cases} \tag{41}$$

because a $\mathcal{C}^1$ trajectory is expected.

- We have successfully tested it on electrical circuit of degree 0 in the semi-implicit case $\theta \in [1/2, 1]$.
- An interesting feature of such $\theta-$method is the energy conserving property that they exhibit for $\theta = 1/2$. We will see in the following section that the scheme can be viewed as a special case of the time-stepping scheme proposed by Pang (2006).

Time stepping scheme for Differential Variational Inequalities (DVI)

In (Pang, 2006), several time-stepping schemes are designed for DVI which are separable in u,

$$\dot{x}(t) = f(t, x(t)) + B(x(t), t)u(t) \quad (42)$$

$$u(t) = \mathrm{SOL}(K, G(t, x(t)) + F(\cdot)) \quad (43)$$

We recall that the second equation means that $u(t) \in K$ is the solution of the following VI

$$(v - u)^T.(G(t, x(t)) + F(u(t))) \geqslant 0, \forall v \in K \quad (44)$$

Two cases are treated with a time-stepping scheme: the Initial Value Problem(IVP) and the Boundary Value Problem(BVP).

Time stepping scheme for DVI. IVP case.

IVP case.

$$\dot{x}(t) = f(t, x(t)) + B(x(t), t)u(t) \quad (45)$$

$$u(t) = \mathrm{SOL}(K, G(t, x(t)) + F(\cdot)) \quad (46)$$

$$x(0) = x_0 \quad (47)$$

The proposed time-stepping method is given as follows

$$x_{k+1} - x_k = h\left[f(t_k, \theta x_{k+1} + (1-\theta)x_k) + B(x_k, t_k)u_{k+1}\right] \quad (48)$$

$$u_{k+1} = \mathrm{SOL}(K, G(t_{k+1}, x_{k+1}) + F(\cdot)) \quad (49)$$

Time stepping scheme for DVI. IVP case.

Explicit scheme $\theta = 0$

An explicit discretization of $\dot{x}$ is realized leading to the one-step non smooth problem

$$x_{k+1} \quad = \quad x_k + h\,[f(t_k, x_k) + B(x_k, t_k)u_{k+1}] \tag{50}$$

where u_{k+1} solves the $VI(K, F_{k+1})$ with

$$F_{k+1}(u) = G(t_{k+1}, h\,[f(t_k, x_k) + B(x_k, t_k)u]) + F(u) \tag{51}$$

Remark

- In the last VI, the value u_{k+1} can be evaluated in explicit way with respect to x_{k+1}.
- It is noteworthy that even in the explicit case, the VI is always solved in a implicit ways, i.e. for x_{k+1} and u_{k+1}.

Semi-implicit scheme

If $\theta \in]0, 1]$, the pair u_{k+1}, x_{k+1} solves the $VI(\mathbb{R}^n \times K, F_{k+1})$ with

$$F_{k+1}(x, u) = \left[\begin{array}{c} x - x_k - h\,[f(t_k, \theta x + (1-\theta)x_k) + B(x_k, t_k)u] \\ G(t_{k+1}, x) + F(u) \end{array}\right] \tag{52}$$

Time stepping scheme for DVI. IVP case.

Convergence results

In (PANG, 2006), the convergence of the semi-implicit case is proved. For that, a continuous piecewise linear function, x^h is built by interpolation of the approximate values x_k,

$$x^h(t) = x_k + \frac{t - t_k}{h}(x_{k+1} - x_k), \forall t \in [t_k, t_k + 1] \qquad (53)$$

and a piecewise constant function u^h is build such that

$$u^h(t) = u_{k+1}, \forall t \in]t_k, t_k + 1] \qquad (54)$$

It is noteworthy that the approximation x^h is constructed as a continuous function rather than u^h may be discontinuous.

Time stepping scheme for DVI. IVP case.

Convergence results

The existence of a subsequence of u_h, x_h denoted by u^{h_ν}, x^{h_ν} such that

- x^{h_ν} converges uniformly to $\hat{x}$ on $[0, T]$
- u^{h_ν} converges weakly to $\hat{u}$ in $\mathcal{L}^2(0, T)$

under the following assumptions:

1. f and G are Lipschitz continuous on $\Omega = [0, T] \times \mathbb{R}^n$,
2. B is a continuous bounded matrix-valued function on Ω,
3. K is closed and convex (not necessarily bounded)
4. F is continuous
5. $SOL(K, q + F) \neq \emptyset$ and convex such that $\forall q \in G(\Omega)$, the following growth condition holds

$$\exists \rho > 0, \sup\{\|u\|, u \in \mathrm{SOL}(K, q + F)\} \leqslant \rho(1 + \|q\|) \qquad (53)$$

This assumption is used to prove that a pair u_{k+1}, x_{k+1} exists for the VI (52). This assumption of the type "growth condition" is quite usual to prove existence of solution of VI through fixed-point theorem (see (Facchinei & Pang, 2003)).

Time stepping scheme for DVI. IVP case.

Convergence results

Furthermore, under either one of the following two conditions:

- $F(u) = Du$ (i.e. linear VI) for some positive semidefinite matrix, D
- $F(u) = \Psi(Eu)$, where Ψ is Lipschitz continuous and $\exists c > 0$ such that

$$\|Eu_{k+1} - E_k\| \leqslant ch \tag{53}$$

all limits $(\hat{x}, \hat{u})$ are weak solutions of the initial-value DVI.
➜ This proof convergence provide us with an existence result for such DVI with a separable in u.
The linear growth condition which is strong assumption in most of practical case can be dropped. In this case, some monotonicity assumption has to be made on F and strong monotonicity assumption on the map $u \mapsto G(t,x) \circ (r + B(t,x)u)$ for all $t \in [0,T], x \in \mathbb{R}^n, r \in \mathbb{R}^n$. We refer to (Pang, 2006) for more details. If $G(x,t) = Cx$, the last assumption means that CB is positive definite.

BVP case

Let us consider now the Boundary value problem with linear boundary function

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t)) + B(x(t), t)u(t) && (54)\\
u(t) &= \mathrm{SOL}(K, G(t, x(t)) + F(\cdot)) && (55)\\
b &= Mx(0) + Nx(T) && (56)
\end{aligned}
$$

The time-stepping proposed by Pang (2006) is as follows :

$$
\begin{aligned}
x_{k+1} - x_k &= h\left[f(t_k, \theta x_{k+1} + (1-\theta)x_k) + B(x_k, t_k)u_{k+1}\right], \quad k \in \{0, \dots, N\\
u_{k+1} &= \mathrm{SOL}(K, G(t_{k+1}, x_{k+1}) + F(\cdot)), \quad k \in \{0, \dots, N-1\}
\end{aligned}
$$

plus the boundary condition

$$
b = Mx_0 + Nx_N \qquad (60)
$$

Comments

The system is henceforth a coupled and large VI for which the numerical solution is not trivial.

Time stepping scheme for DVI. BVP case

Convergence results

The existence of the discrete time-trajectory is ensured under the following assumption :

1. F monotone and VI solutions have linear growth
2. the map $u \mapsto G(t,x) \circ (r + B(t,x)u)$ is strongly monotone
3. $M + N$ is non singular and satisfies

$$\exp(T\psi_x) < 1 + \frac{1}{\|(M+N)^{-1}N\|}$$

where x ¿ 0 is a constant derived from problem data.

The convergence of the discrete time trajectory is proved if F is linear.

Time stepping scheme for Differential Variational Inequalities (DVI)

General remarks

- The time–stepping scheme can be viewed as extension of the DCS, the UDI and the Moreau's catching up algorithm.
- But, the scheme is more a mathematical discretization rather a numerical method. In practice, the numerical solution of a VI is difficult to obtain when the set K is unstructured.
- The case K is polyhedral is equivalent to a DCS.

Thank you for your attention.

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