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A categorical approach to time representation first study on qualitative aspects

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ABSTRACT: *The qualitative time representation formalisms are considered from the viewpoint of category theory. The representation of a temporal situation can be expressed as a graph and the relationship holding between that graph and others (imprecise or coarser) views of the same situation are expressed as morphisms. These categorical structures are expected to be combinable with other aspects of knowledge representation providing a framework for the integration of temporal representation tools and formalisms with other areas of knowledge representation.*

KEYWORDS: *Category theory, time representation, temporal granularity, interval algebra.*

Time and space representation are only one aspect of knowledge representation. It is thus useful to place them in a wider context. Category theory which is widely used in programming language semantics has been introduced in knowledge representation [AÏTK93] in order to account for the relation of approximation between, on the one hand, a knowledge base and the modeled domain, and on the other, the many achievements of that knowledge base. This notion of approximation generalizes that of interpretation of classical logic in that it allows to take into account that a representation can get closer — instead of exactly correspond — to the modeled domain. It seems that this analysis can also be applied to time representation which combine several internal approximation mechanisms: qualitative interpretation, discretizing and weakening (see figure 1).

Moreover, knowledge representation formalisms are more and more specialized. However, it is expected that they can be combined in order to represent a complex domain (e.g. for adding temporal extension to objects represented as Ψ-terms [AÏTK93]) in such a way that their mathematical properties are preserved. Once this has been achieved, the attention can turn to the interaction between the formalisms in a particular application.

Category theory [BARR90, PIER91, BORC94] deals with objects (characterized by their structure) and morphisms (preserving the structure). This immediately expresses the notion of approximation of one structure by another. Nonetheless, one of the advantages of category theory is its ability to model the interactions between categories through a variety of operations. These operations can be used to combine the independently designed knowledge representation schemes as soon as they are characterized as categories.

The present communication is a first report on the rephrasing, in categorical terms, of the tools for qualitative time and space representation. The theoretical advantage expected from such a characterization is the expression of the relationship between a qualitative representation of a temporal domain and:

1) a less precise qualitative representation of the same domain;
2) a "deductively closed" qualitative representation of the same domain;

3) a coarser qualitative representation of the same domain.

The final aim of this work is the use of category theory for taking into account the relationship between the qualitative representation of time and its quantitative interpretation. As a matter of fact, there is an immediate relationship between quantitative representations at different scales, a far from immediate relationship between discrete quantitative representations and an expected relationship between granular qualitative representation and discrete quantitative representation (see figure 1). Such a characterization would result in a standard notion of temporal representation combinable with any other knowledge representation formalism.

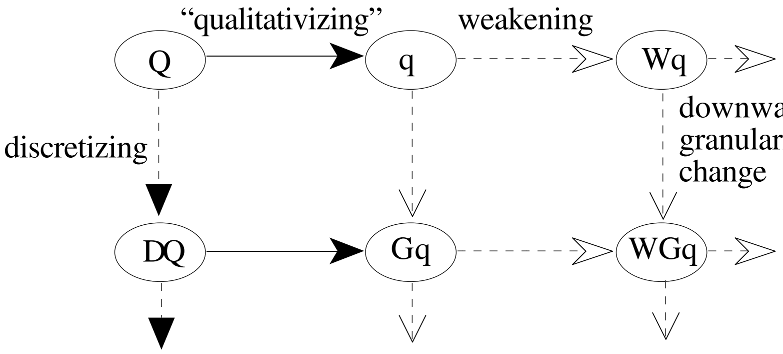

Figure 1. The relationship existing between different models of the same domain. It depends on the nature of the representation (Quantitative/qualitative), its degree of simplification (Discrete — Granular) and the expressiveness of the language used (Weak). Dashed arrows represent operations which can be achieved in many ways and pending arrows those which can be applied to some type of representation providing a new representation of the same type. Note that arrows and nodes are not categorical notation.

The remainder is a first attempt to settle the qualitative temporal (resp. spatial) representation of some situation as an object and the relationship it enjoys with other representations as morphisms. It can also be understood as an introduction by the example to categorical notions and their applicability to artificial intelligence research. This first picture is under construction and by no means complete: some of the concepts will have to be refined and others modified.

This short report is not self-contained: it introduces only minimal definitions (noted meta-definitions) for categorical notions and does not introduce notions on qualitative time and space representation. A first section introduces the qualitative interval representation as an object and considers various kind of morphisms (points 1 and 2 above). Section 2 introduces qualitative granularity as a new kind of morphism (point 3 above). For reason of space, section 3 and 4 are reduced and presents some other potential application of category theory to the field.

1.Approximation in symbolic time representations

Qualitative representation of time (e.g. the interval algebra [ALLE83]) represents the temporal situation as a complete directed graph whose nodes are temporal entities (e.g. intervals) and edges the relationship between them. The set of temporal relationships considered can vary with the formalism and is noted Γ here (e.g. A_{13} for Allen relationships). These graphs are considered as the objects of a category (§1.1). The notion of approximation can be modeled through various kind of morphisms which are considered here (§1.2 and 1.3).

1.1.Time graphs

The representation of a temporal situation can be made of a graph (called Γ-graph) whose nodes are the temporal entities and whose edges are labeled with a sub-set of a set Γ.

DEFINITION (Γ-graph): A Γ-graph A is a complete directed graph made of a set N^A of nodes and a set E^A of labeled edges $<n_1,l,n_2>$ such that $n_1,n_2 \in N^A$, $n_1 \neq n_2$, $l \subseteq \Gamma$ and $l \neq \emptyset$. Moreover, for each $n_1,n_2 \in N^A$ such that $n_1 \neq n_2$ there exists exactly one $<n_1,l,n_2> \in E^A$.

Such graphs are interpreted as a temporal situation in which the nodes are temporal intervals (places in time) and the labels carry the set of possible temporal relations between the two connected intervals. The constraint about non emptiness aims at ensuring that between any two intervals there is always one possible temporal relation (as it is in time).

A less constrained definition of temporal graphs (made of the same set N^A of and a set E^A of edges such that for some distinct nodes n_1 and n_2 in N^A there exist several edges and for others there exists no edge at all) can be straightforwardly normalized by having the only edge between the nodes being labeled by the intersection of the labels of all the initial edges and Γ.

POSTULATE (temporal representation): the modeled domain can be represented by a Γ-graph.

Any one will admit that this postulate is assumed by anyone using qualitative time representation.

1.2. More precise representation

A more precise representation of the same situation is understood as a representation with more nodes and smaller labels on the edges. This is expressed through the notion of a Γ-χ-morphism.

DEFINITION (Γ-χ-morphism): A Γ-χ-morphism between two Γ-graphs $A=<N^A,E^A>$ and $B=<N^B,E^B>$ is an injective map γ such that:

- $\forall n \in N^A$, $\gamma(n) \in N^B$, and
- $\forall <n_1,l,n_2> \in E^A$, $\gamma(<n_1,l,n_2>)=<\gamma(n_1),l',\gamma(n_2)>$ with $l' \subseteq l$.

As defined the relationship between Γ-graphs introduced by Γ-χ-morphisms is reflexive. Note that the definition of graphs and morphisms could have been carried by a fixed set of nodes N thus only the node labels would have changed through morphisms. This is not the case and henceforth, the "operation" carried out by the morphism consist in (1) restricting the existing labels in the domain and (2) adding new nodes and edges. Below this is simply called restriction.

CONSEQUENCE 1: For each Γ-graph A there exists a Γ-χ-endomorphism id_A, such that:

- $\forall n \in N$, $id_A(n)=n$, and
- $\forall e \in E$, $id_A(e)=e$ (with systematically $l'=l$).

META-DEFINITION (Category): The structure made of:

1) a collection of objects;
2) a collection of arrows f:A→B (or morphisms) from an object A (domain) to an object B (co-domain);
3) a composition operation "o" assigning to each pair of arrows f:A→B and g:B→C a composite arrow g o f:A→C.

such that:

a) for each object A there exists an identity arrow: id_A:A→A satisfying the identity laws: for any arrow f:A→B, id_B o f = f and f o id_A = f.
b) the composition is associative: for any arrows f:A→B, g:B→C and h:C→D, h o (g o f) = (h o g) o f.

is called a category.

We now prove that Γ-graphs and Γ-χ-morphisms form a category.

PROPOSITION 2: The structure made of:

1) a collection of Γ-graphs (called objects);
2) a collection of Γ-χ-morphism between these objects among which are the id_A for each object A;
3) an operator "o" corresponding to function composition;

is a category (called Γ-χ-category).

proof.

a) for any arrow $f:A \rightarrow B$, id_B o f = f and f o id_A = f (property of the identity function);
b) for any arrows $f:A \rightarrow B$, $g:B \rightarrow C$ and $h:C \rightarrow D$, h o (g o f) = (h o g) o f (since the composition of label restriction/node addition is possible and there is only one way to restrict node labels and add new nodes such that a graph corresponds to another). ◊

This proposition only means that:

- the graph A is an empty restriction of itself, so the restriction of A to B is the same restriction as that from A to A composed with that from A to B;
- there is only one way to restrict A to D which does not depend on the intermediate steps.

META-DEFINITION (initial object, terminal object): An object 0 is called an initial object if, for every object A, there is exactly one arrow from 0 to A; an object 1 is called a terminal object if, for every object A, there is exactly one arrow from A to 1.

PROPOSITION 3: There is an initial object in the Γ-χ-category, the one with no nodes.

proof. for such an object, there is always a Γ-χ-morphism to any other object and since the restriction is unique, this morphism is also unique. ◊

If we consider the graphs carried by a set of nodes *N*, then there also is an initial object which is the graph with all edges labeled with Γ.

In temporal representation, the intuition behind the initial object is the representation which tells nothing about the relationship between the objects. At the opposite, we would like to consider the terminal object as the domain to be modeled. So, each Γ-graph would be an approximation of that domain. However, this has not been considered here (because non correct graph, with regard to that domain, are allowed). A construction meeting that intuition can be found in [AÏTK93]. It consists in restricting the category to these graphs which are an approximation of the modeled domain representation (the category generated by it). Then the graph matching exactly the reality would be terminal.

In addition, we can also define these objects which are not the domain of any morphism (but their identity arrow). Such objects only have singleton labels (if we do not consider any addition of nodes). They represent an instantiated situation which could be the modeled domain (which is such an object from the postulate). As a matter of fact, all such graphs which are approximated by a particular Γ-graph exactly correspond to the models of (i.e. the possible real situations represented by) this last graph.

1.3.Constraint resolution

Preciseness seems a natural notion for Γ-graphs. However, qualitative temporal systems are provided with a deductive operation: the application of transitivity through Allen's composition table [ALLE83]. They allow the reduction of the edge labels (thus leading to a more precise Γ-graph). The deduction made is first shown as a morphism and then the intuition that the result of deduction is more precise is established. So, a new kind of morphism: "is deductible through constraint propagation from" is introduced.

DEFINITION (Γ-δ-morphism): A Γ-δ-morphism between two Γ-graphs $A=\langle N^A,E^A\rangle$ and $B=\langle N^B,E^B\rangle$ is an injective map γ such that:

- $\forall n\in N^A$, $\gamma(n)\in N^B$, and
- $\forall \langle n_1,l,n_2\rangle\in E^A$, $\gamma(\langle n_1,l,n_2\rangle)=\langle\gamma(n_1),l',\gamma(n_2)\rangle$ such that l' is deducible through constraint propagation from A.

As expected, what is deduced is a more precise graph.

CONSEQUENCE 4: A Γ-δ-morphism is a Γ-χ-morphism (since the deduction process only reduces the labels).

CONSEQUENCE 5: For each Γ-graph A there exists a Γ-δ-endomorphism id_A, such that:

- $\forall n\in N$, $id_A(n)=n$, and
- $\forall e\in E$, $id_A(e)=e$ (since the labels are deducible from themselves).

PROPOSITION 6: The structure made of:

1) a collection of Γ-graphs (called objects);
2) a collection of Γ-δ-morphism between these objects among which are the id_A for each object A;
3) an operator "o" corresponding to function composition;

is a category (called Γ-δ-category).

proof.

a) for any arrow $f:A\rightarrow B$, id_B o f = f and f o id_A = f (since no reduction applied before or after a reduction is always this last reduction);
b) for any arrows $f:A\rightarrow B$, $g:B\rightarrow C$ and $h:C\rightarrow D$, h o (g o f) = (h o g) o f (since, the deduction — by successive application of the transitivity table — can be composed and there is only one way to reduce the graph A directly to the graph D). ◊

This is also easily obtained by adapting the proof that deduction systems are categories (in which objects are formulas and arrows are proofs) [BARR90] to the constraint satisfaction case:

- for any graph, there is an empty proof from itself (id_A: $A\rightarrow A$), and if there is a proof p of A from B, then p o id_B and id_A o p are the same proof p;
- if there is a proof p of A from B, a proof q of B from C and a proof r from C to D, then, the proofs obtained by chaining r o (q o p) and (r o q) o p are the same.

An attempt can be made for normalizing the graphs such that each graph is represented by its more reduced form through constraint propagation. This can be achieved with the help of a functor mapping each graph to its normal form (provided it is unique). The resulting category would be a sub-category of the initial one.

2.Granularity

The granular representation has been introduced in [EUZE93, 94]. It represents a qualitative temporal situation in the same way as any original qualitative language. However, this representation can differ from the "exact" one since it considers that, under a coarser grain, some objects do not exist anymore (thus changing the labels of the remaining edges). These changes between two different granular representations are defined by an operator $\downarrow$ on the set of relations between temporal entities [EUZE94].

Thus, a granular representation is expressed as a Γ-graph exactly as the exact representation and the relationship between representations from different granularities is taken into account as a new kind of morphism (Γ-γ-morphisms). These morphisms correspond to downward granularity change, thus going from a coarse view of the situation to a finer one (with more objects).

DEFINITION (Γ-γ-morphism): A Γ-γ-morphism between two Γ-graphs $A=<N^A,E^A>$ and $B=<N^B,E^B>$ is an injective map γ such that:

- $\forall n\in N^A$, $\gamma(n)\in N^B$, and
- $\forall <n_1,l,n_2>\in E^A$, $\gamma(<n_1,l,n_2>)=<\gamma(n_1),l',\gamma(n_2)>$ such that $l'\subseteq\downarrow l$.

Γ-morphisms do not prevent new nodes from appearing in the image Γ-graph; this exactly models the possible vanishing of objects through granularity change. It is also possible that two distinct objects in the domain have the same image; this has an interpretation which is not developed here.

CONSEQUENCE 7: A Γ-χ-morphism is a Γ-γ-morphism (since, by [EUZE95, property (1)], $l\subseteq\downarrow l$ and l' can be chosen in l).

CONSEQUENCE 8: For each Γ-graph A there exists a Γ-γ-endomorphism id_A, such that:

- $\forall n\in N$, $id_A(n)=n$, and
- $\forall e\in E$, $id_A(e)=e$ (since, by [EUZE95, property (1)], $l\subseteq\downarrow l$.).

PROPOSITION 9: The structure made of:

1) A collection of Γ-graphs (called objects);
2) A collection of Γ-γ-morphism between these objects (called arrows ${}^A\downarrow_B$ of domain A and co-domain B) among which are the id_A for each object A;
3) An operator "o" corresponding to function composition (the symbol "·" used in [EUZE93, 94, 95]);

is a category (called Γ-γ-category).

proof.

a) id satisfies the identity laws: $id_B \circ {}^A\downarrow_B = {}^A\downarrow_B$ and ${}^A\downarrow_B \circ id_A = {}^A\downarrow_B$ (property of the identity function);
b) for any arrows ${}^A\downarrow_B$:A→B and ${}^B\downarrow_C$:B→C and ${}^C\downarrow_D$:C→D, ${}^C\downarrow_D \circ ({}^B\downarrow_C \circ {}^A\downarrow_B) = ({}^C\downarrow_D \circ {}^B\downarrow_C) \circ {}^A\downarrow_B$ is obviously true owing to [EUZE95, property (5)]. ◊

There exists a particular Γ-graph which represents the empty world (so coarse that no object in the modeled domain is relevant). This object is initial.

PROPOSITION 10: Γ-γ-categories have an initial object 0 which is the Γ-graph <Ø,Ø>.

proof. obvious, through an empty Γ-γ-morphism from 0 to any other Γ-graph and the impossibility of having a morphism from a non empty graph to 0. ◊

Like in the previous section, there is no terminal object for Γ-γ-categories. This is caused by the possibly infinite chain of new nodes that the granularity changes can add to the graphs. However, if the categories are restricted to be generated by a particular domain restriction then the graph representing the modeled domain should be a terminal object.

An important question would be if granularity changes and deduction can be freely interleaved. However, we proved in [EUZE93, 94] that this is not generally the case.

3.Extension to spatial formalisms

The success of qualitative time representation has led to extensions toward spatial qualitative representations. The more simple are those which considers the space as a Cartesian product of linear representation (such as the one used for time [GÜSG89]) or those which consider a space topology as a weakening of the time algebra [EGEN92, RAND92]. These transformations of the formalism has been used in [EUZE94, 95] for extending in a straightforward fashion the results of temporal granularity to these space representations. These results can also be expressed in categorical terms (respectively as products of categories and functors).

4.Towards quantitative models

All the temporal representations presented above are supported by an idea of a measurable reality. As a matter of fact, a set of the real number intervals is the natural interpretation of these notions. Thus further work should draw the connection between the Γ-graphs and their possible models. This can be achieved in the context of category theory (more simply than what have been done for λ-calculus for instance). The general mechanism for doing this is expressed as "A mathematical theory — corresponding roughly to the definition of a class of mathematical objects — can be usefully regarded as a category of a certain kind, and a model of that theory — one of those objects — as a set-valued functor from that category which preserves the structure." [Lawvere quoted in PIER91]. Once this has been done it must be proved (for any kind of Γ-morphism) that for any A and B Γ-graphs, there exists a Γ-morphism $\gamma{:}A \rightarrow B$ if and only if the set of models of A is included in that of B. This has been used, for instance, in [LI95].

A dual way would consist in beginning with the domain to be modeled as the basic category and defining morphisms from that domain (intervals of real numbers) to the Γ-graphs. The scaling should as well be characterized in this fashion (see figure 1) and be approximated by granularity.

5.Final note

Seeing the many possible representations of a temporal situation as approximations in the same framework is the contribution of category theory to time representation. It allows to conceive these representations in a uniform way.

This short presentation is subject to debate and some problems will have to be solved before to turn to a full account of time representations including the quantitative approach and/as model theory. For instance, as suggested by one of the reviewers, the treatment of graphs with empty labels is not sound. Such graphs have been rejected here under the justification of avoiding inconsistent graphs. However, there can be inconsistent graphs with no empty label. This is very problematic if one thinks that graphs with empty labels can be deduced through constraint propagation from a valid graph. The rejection of graphs with empty labels has been retained here because such graphs are obviously meaningless. The interpretation of morphisms being approximation, it would be peculiar to consider that the description of any real situation (but that with nothing to represent) is an approximation of something meaningless. Ruling out graphs with empty labels allows to preserve an (informal) idea of potential minimality of the most accurate representation. However, the current situation is not satisfactory.

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