SOME NEWS ON AUGMENTED LAGRANGIAN METHODS

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(mostly) joint work with

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Outline

- The augmented Lagrangian method
  (the method of multipliers)
- Previous rate of convergence results:
  dual Q-linear and primal R-linear if
  - LICQ, strict complementarity, SOSC
- New rate of convergence results:
  primal-dual Q-linear and primal Q-linear if
  - SOSC only (no SC, no LICQ)
- Flavor of the analysis
  (Newtonian analysis of non-Newtonian method!)
- Numerical comparisons (6 solvers, MacMPEC)

The augmented Lagrangian algorithm

For the problem

$$\min\ f(x) \quad \text{s.t.} \quad h(x) = 0,\ g(x) \le 0,$$

given the multiplier estimates (λ^k, μ^k),
the method of multipliers computes x^{k+1} by

$$\begin{aligned} \min_x\ L_{c_k}(x, \lambda^k, \mu^k) := \quad & f(x) + \langle \lambda^k, h(x) \rangle + \frac{c_k}{2} \|h(x)\|^2 \\ & + \frac{1}{2c_k} \Big(\| \max\{0, \mu^k + c_k g(x)\} \|^2 - \|\mu^k\|^2 \Big) \end{aligned}$$

$$\lambda^{k+1} = \lambda^k + c_k h(x^{k+1}), \quad \mu^{k+1} = \max\Big\{0, \mu^k + c_k g(x^{k+1})\Big\}.$$

LANCELOT and ALGENCAN software; Hestenes, Powell, Rockafellar, Bertsekas, ...

The augmented Lagrangian algorithm

More generally, for the problem

$$\min\ f(x) \quad \text{s.t.} \quad h(x)=0,\ g(x)\le 0,\ Ax\le a,$$

the linear constraints are handled directly:

$$\min_{Ax\le a}\ L_{c_k}(x,\lambda^k,\mu^k) := \ f(x)+\langle\lambda^k,h(x)\rangle+\frac{c_k}{2}\|h(x)\|^2 + \frac{1}{2c_k}\left(\|\max\{0,\mu^k+c_k g(x)\}\|^2-\|\mu^k\|^2\right)$$

$$\lambda^{k+1}=\lambda^k+c_k h(x^{k+1}),\qquad \mu^{k+1}=\max\left\{0,\mu^k+c_k g(x^{k+1})\right\}.$$

The augmented Lagrangian algorithm

To shorten notation, define

$$Q = \{\nu \in \mathbf{R}^m \mid \nu_i \in \mathbf{R},\ i = 1, \dots, l;\ \nu_i \geq 0,\ i = l+1, \dots, m\},$$

then the problem becomes

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \in Q^\circ,$$

and x^{k+1} is given by

$$\min_x\ L_{c_k}(x, \mu^k) := f(x) + \frac{1}{2c_k}\left(\left\|\Pi_Q\left(\mu^k + c_k g(x)\right)\right\|^2 - \|\mu^k\|^2\right),$$

with the new multiplier estimates given by

$$\mu^{k+1} = \Pi_Q\left(\mu^k + c_k g(x^{k+1})\right).$$

Standard rate of convergence statement

$$\|\mu^{k+1} - \bar{\mu}\| \leq \frac{q}{c_k}\|\mu^k - \bar{\mu}\| \quad \text{(dual } Q\text{-linear)}$$

$$\|x^{k+1} - \bar{x}\| \leq \frac{q}{c_k}\|\mu^k - \bar{\mu}\| \quad \text{(primal } R\text{-linear)}$$

if μ^0 close to $\bar{\mu}$, $c_k > \bar{c}$, and

- LICQ: active constraints gradients linearly idependent ($\bar{\mu}$ is unique),
- Strict complementarity: $\bar{\mu}_i > 0$ for all active inequality constraints,
- SOSC: $\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\mu})d, d \right\rangle > 0 \quad \forall d \in C(\bar{x}) \setminus \{0\}$.

Some considerations

- These assumptions are strong
  (not much more could possibly be asked...)
- Not quite clear what for are LICQ and strict complementarity...
  (other than to employ certain proof techniques)
- SOSC is needed indeed...
  (gives local coercivity of $L_{c_k}(\cdot, \mu^k)$, which ensures the existence of the minimizer x^{k+1})
- Perhaps SOSC is all that is really needed? Yes!

New rate of convergence results

If μ^0 close to $\bar{\mu} \in M(\bar{x})$ with SOSC at $\bar{\mu}$, $c_k > \bar{c}$, then

$$\{(x^k, \mu^k)\} \to (\bar{x}, \hat{\mu}), \quad \hat{\mu} \in M(\bar{x}),$$

$$\frac{\|(x^{k+1}, \mu^{k+1}) - (\bar{x}, \hat{\mu})\|}{\|(x^k, \mu^k) - (\bar{x}, \hat{\mu})\|} \leq \frac{q}{c_k}$$ (primal-dual Q-linear)

$$\frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} \leq \theta \, \frac{\|(x^{k+1}, \mu^{k+1}) - (\bar{x}, \hat{\mu})\|}{\|(x^k, \mu^k) - (\bar{x}, \hat{\mu})\|}$$ (primal Q-linear)

No LICQ, no strict complementarity!

Flavor of the analysis

The method does the following

$$\left\| \frac{\partial L_{c_k}}{\partial x}(x^{k+1}, \mu^k) \right\| \leq \epsilon_k, \qquad \mu^{k+1} = \Pi_Q(\mu^k + c_k g(x^{k+1})).$$

Observe

$$\begin{aligned} \frac{\partial L_{c_k}}{\partial x}(x^{k+1}, \mu^k) &= f'(x^{k+1}) + (g'(x^{k+1}))^\top \Pi_Q(\mu^k + c_k g(x^{k+1})) \\ &= \frac{\partial L}{\partial x}(x^{k+1}, \mu^{k+1}), \end{aligned}$$

so that

$$0 \in \frac{\partial L}{\partial x}(x^{k+1}, \mu^{k+1}) + B(0, \epsilon_k).$$

Flavor of the analysis

Therefore, the method is

$$0 \in \frac{\partial L}{\partial x}(x^{k+1}, \mu^{k+1}) + B(0, \epsilon_k), \qquad \mu^{k+1} = \Pi_Q(\mu^k + c_k g(x^{k+1})).$$

From the projection condition,

$$\mu^k + c_k g(x^{k+1}) - \mu^{k+1} \in N_Q(\mu^{k+1}).$$

And, puting together, the iteration satisfies the GE

$$0 \in \begin{bmatrix} \dfrac{\partial L}{\partial x}(x^{k+1}, \mu^{k+1}) \\ -g(x^{k+1}) + \dfrac{1}{c_k}(\mu^{k+1} - \mu^k) \end{bmatrix} + \begin{bmatrix} B(0, \epsilon_k) \\ N_Q(\mu^{k+1}) \end{bmatrix}.$$

Flavor of the analysis

Thus, the method is

$$0 \in \begin{bmatrix} \dfrac{\partial L}{\partial x}(x^{k+1}, \mu^{k+1}) \\ -g(x^{k+1}) + \dfrac{1}{c_k}(\mu^{k+1} - \mu^k) \end{bmatrix} + \begin{bmatrix} B(0, \epsilon_k) \\ N_Q(\mu^{k+1}) \end{bmatrix}.$$

In other words, (x^{k+1}, μ^{k+1}) solves (perturbed) GE

$$0 \in \begin{bmatrix} \dfrac{\partial L}{\partial x}(x, \mu) \\ -g(x) \end{bmatrix} + \begin{bmatrix} 0 \\ N_Q(\mu) \end{bmatrix} + \begin{bmatrix} B(0, \epsilon_k) \\ \dfrac{1}{c_k}(\mu^{k+1} - \mu^k) \end{bmatrix},$$

i.e., KKT optimality conditions + perturbation

For the problem

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \in\ Q^{\circ},\ Ax \leq a,$$

we do

$$\min\ L_{c_k}(x, \mu^k) \quad \text{s.t.} \quad Ax \leq a,$$

$$\mu^{k+1} = \Pi_Q \left(\mu^k + c_k g(x^{k+1}) \right).$$

This means

$$0 \in \frac{\partial L_{c_k}}{\partial x}(x^{k+1}, \mu^k) + A^{\top} \nu^{k+1} + B(0, \epsilon_k),$$

$$Ax^{k+1} - a \leq 0, \quad \nu^{k+1} \geq 0, \quad \langle \nu^{k+1}, Ax^{k+1} - a \rangle = 0,$$

$$g(x^{k+1}) - \frac{1}{c_k}(\mu^{k+1} - \mu^k) \in N_Q(\mu^{k+1}).$$

Thus, for the problem

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \in Q^\circ,\ Ax \le a,$$

$(x^{k+1}, \mu^{k+1}, \nu^{k+1})$ solves (perturbed) GE

$$0 \in \begin{bmatrix} \dfrac{\partial L}{\partial x}(x, \mu, \nu) \\ -g(x) \\ -(Ax - a) \end{bmatrix} + \begin{bmatrix} 0 \\ N_Q(\mu) \\ N_{\mathbf{R}^s_+}(\nu) \end{bmatrix} + \begin{bmatrix} B(0, \epsilon_k) \\ \dfrac{1}{c_k}(\mu^{k+1} - \mu^k) \\ 0 \end{bmatrix},$$

i.e., KKT conditions + (same) perturbation

Flavor of the analysis

Analysis of perturbed generalized equations

$$\mathrm{SOL}(p) = \{(x, \mu) \mid 0 \in G(x, \mu) + p + N(x, \mu)\}.$$

Under SOSC, the KKT GE is upper-Lipschitzian

$$\mathrm{SOL}(p) \cap \Delta \subset \mathrm{SOL}(0) + \tau_1 \|p\| B(0, 1) \qquad \text{for } p \text{ “small”}.$$

For KKT GE, the latter property is equivalent to

- Local error bound given by KKT residual
- The multiplier $\bar{\mu} \in M(\bar{x})$ is noncritical

A. Izmailov and M. Solodov, Math. Programming, Vol. 133, 2012.

Upper-Lipschitzian behavior of perturbed KKT GE

$$0 \in \begin{bmatrix} \dfrac{\partial L}{\partial x}(x,\mu) \\ -g(x) \end{bmatrix} + \begin{bmatrix} 0 \\ N_Q(\mu) \end{bmatrix} + p$$

$$\Updownarrow$$

$$\mathrm{dist}((x,\mu),\mathrm{SOL}) \le \beta \left\| \begin{bmatrix} \dfrac{\partial L}{\partial x}(x,\mu) \\ \mu - \Pi_Q(\mu + g(x)) \end{bmatrix} \right\|$$

$$\Updownarrow$$

$d = 0$ is the unique stationary point of

$$\min \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x},\bar{\mu})d, d \right\rangle \quad \text{s.t.} \quad d \in C(\bar{x})$$

Flavor of the analysis

is Newtonian:

- Prove existence of solutions (x^{k+1}, μ^{k+1}) of GE, given any μ^k close to $\bar{\mu}$;
- Prove that the primal-dual step is "small" (so that (x^{k+1}, μ^{k+1}) stays in the neighborhood):

$$\|(x^{k+1}, \mu^{k+1}) - (x^k, \mu^k)\| \le \tau_2 \mathrm{dist}((x^k, \mu^k), \mathrm{SOL});$$

- Then upper-Lipschitzian property gives

$$\mathrm{dist}((x^{k+1}, \mu^{k+1}), \mathrm{SOL}) \le \tau_1 \|p^k\| \le \ldots.$$

Somewhat related to Stabilized SQP, D. Fernández & M. Solodov, Math. Programming, 2010.

Flavor of the analysis

So we have (simplifying with $\epsilon_k = 0$)

$$\mathrm{dist}((x^{k+1}, \mu^{k+1}), \mathrm{SOL}) \le \frac{\tau_1}{c_k} \|\mu^{k+1} - \mu^k\|,$$

and

$$\|(x^{k+1}, \mu^{k+1}) - (x^k, \mu^k)\| \le \tau_2 \mathrm{dist}((x^k, \mu^k), \mathrm{SOL}).$$

From the latter,

$$\|\mu^{k+1} - \mu^k\| \le \tau_2 \mathrm{dist}((x^k, \mu^k), \mathrm{SOL}),$$

and hence,

$$\mathrm{dist}((x^{k+1}, \mu^{k+1}), \mathrm{SOL}) \le \frac{\tau_1 \tau_2}{c_k} \mathrm{dist}((x^k, \mu^k), \mathrm{SOL})$$

Finalizing,

$$\text{dist}((x^{k+1}, \mu^{k+1}), \text{SOL}) \le \frac{\tau_1 \tau_2}{c_k} \text{dist}((x^k, \mu^k), \text{SOL})$$

gives Q-linear convergence of the distance to the primal-dual solution set.

Further analysis shows

- Convergence of $\{(x^k, \mu^k)\}$ to $(\bar{x}, \hat{\mu})$, $\hat{\mu} \in M(\bar{x})$;
- Q-linear rate of convergence of $\{x^k\}$.

Boundedness of the penalty parameters

In ALGENCAN, for $\theta \in (0,1)$ and $\rho > 1$,

$$c_{k+1} = \begin{cases} c_k, & \text{if } \sigma(x^{k+1}, \mu^{k+1}) \le \theta \sigma(x^k, \mu^k), \qquad (*) \\ \rho c_k, & \text{otherwise,} \end{cases}$$

where

$$\sigma(x,\mu) = \left\| \begin{bmatrix} \dfrac{\partial L}{\partial x}(x,\mu) \\ \mu - \Pi_Q(\mu + g(x)) \end{bmatrix} \right\|$$

As a by-product of our analysis,

rate of $\sigma(x^k, \mu^k) \to 0$ is inversely proportionate to c_k.

Hence, the linear test (*) holds once c_k large enough.

On problems with complementarity constraints

$$\min \; f(x) \quad \text{s.t.} \quad H(x) \geq 0, \; G(x) \geq 0, \; \langle H(x), G(x) \rangle = 0$$

- no MFCQ (or LICQ) at any feasible point (hence, multiplier set always unbounded)
- RCPLD (and even CPLD) can hold... but only for artificial examples
- Augmented Lagrangian methods need no CQs... Attractive for MPCC? At least worth a try!

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joint work with A. Izmailov and E. Uskov

Theoretical global convergence on MPCC

If MPCC-LICQ holds at a feasible primal accumulation point $\bar{x}$,

- Then $\bar{x}$ is strongly stationary (KKT) when a certain part of the dual sequence is bounded;
- Otherwise, it is at least C-stationary.

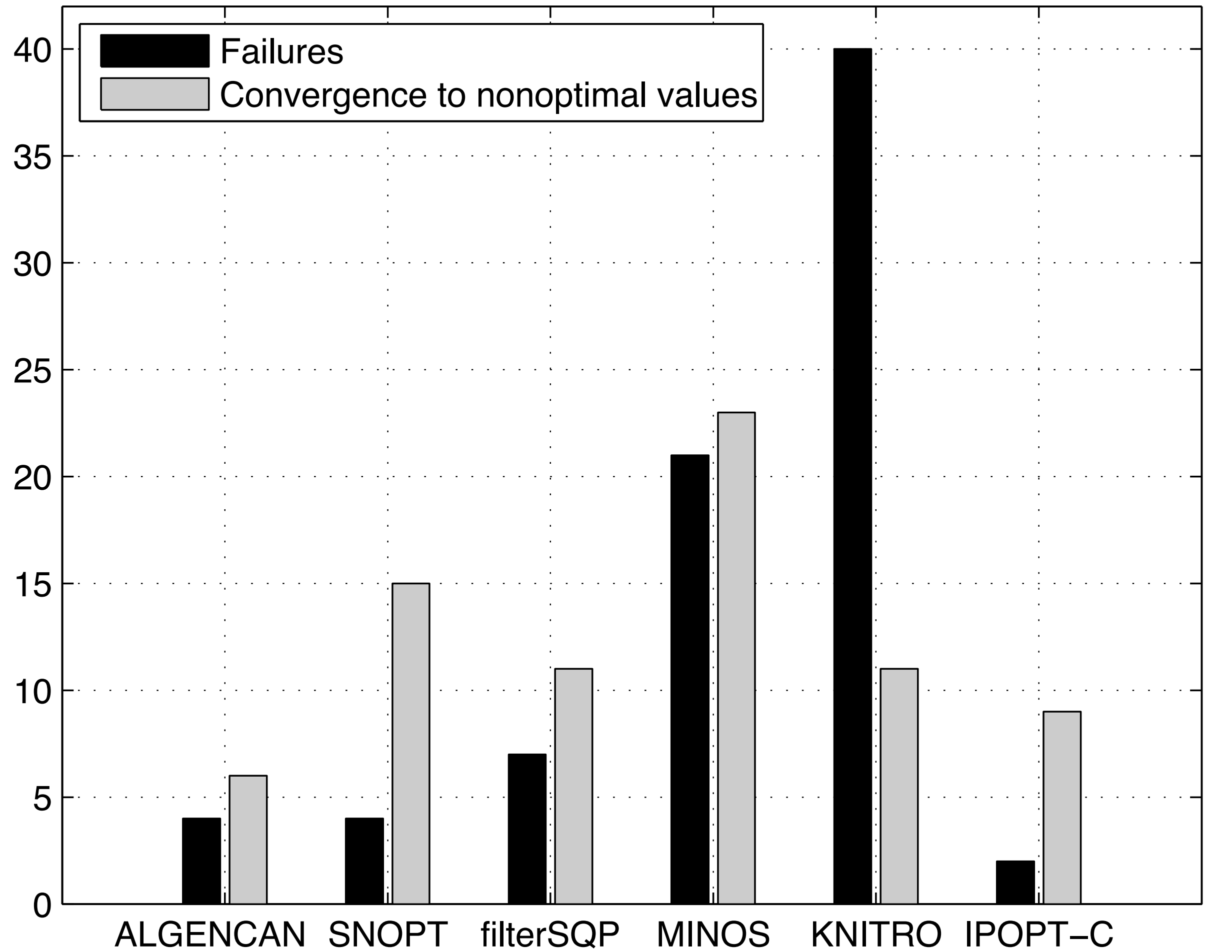
Considerations:

- Ideally, want strong stationarity. However...
- Seems impossible with usual NLP tools (without expensive combinatorial overhead)
- The stated is competitive with state-of-the-art (SQP can converge to arbitrary feasible point)
- In practice, augmented Lagrangian usually achieves strong stationarity.
  - many problems do not have C-stationary non-strongly stationary points
  - augmented Lagrangian has an intrinsic dual stabilization property which tends to keep dual sequences bounded (even for MPCC!)

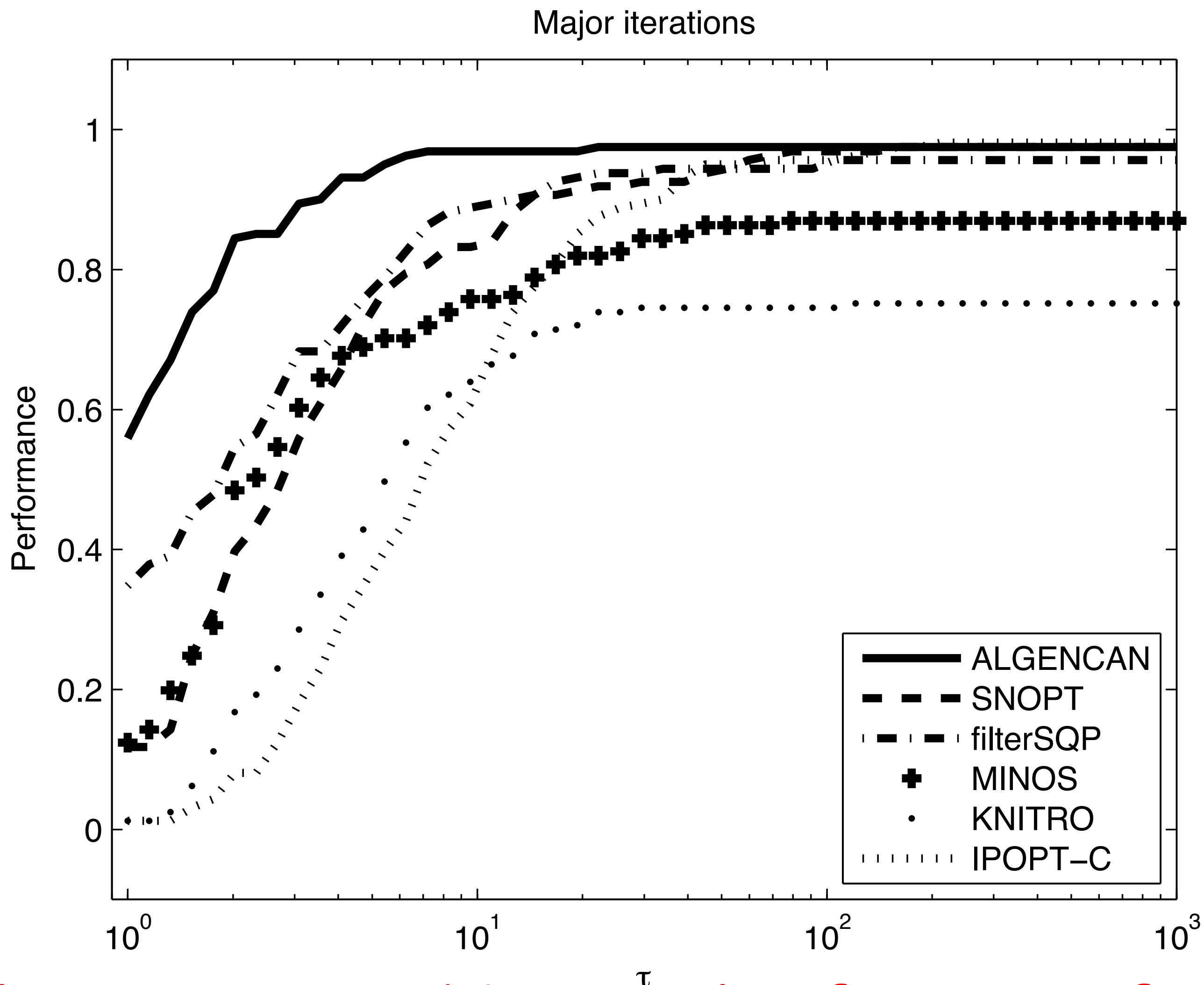
Comparisons on MacMPEC

6 solvers:

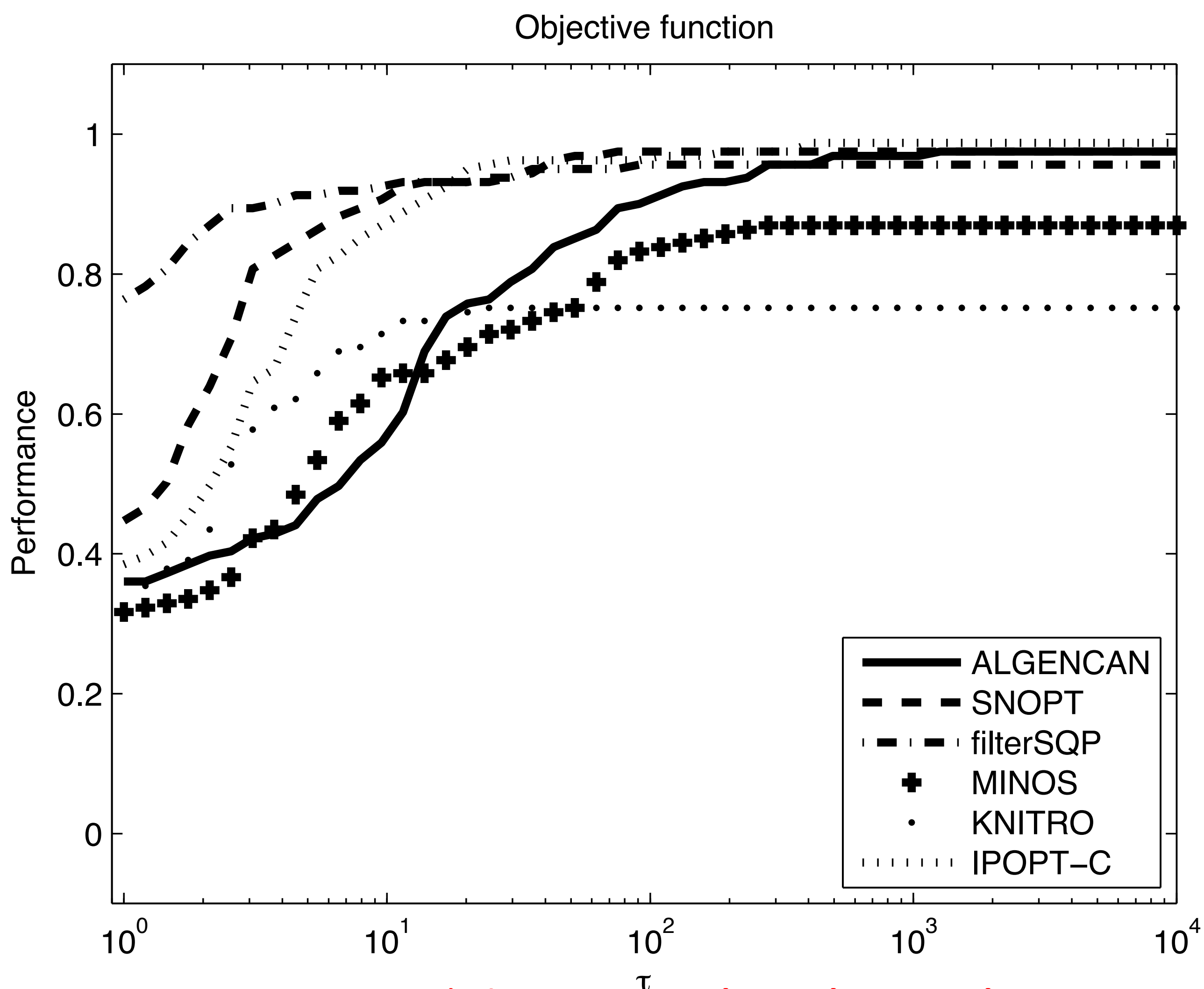
- ALGENCAN (augmented Lagrangian)
- SNOPT (elastic mode linesearch SQP)
- filterSQP (trust-region filter SQP)
- MINOS (linearized augmented Lagrangian)
- KNITRO (interior points, with MPCC option)
- IPOPT-C (interior points, with MPCC option)



Augmented Lagrangian is robust;
computes good solutions!



Confirms augmented Lagrangian fast rate of convergence results!



However, augmented Lagrangian iterations are expensive...

Conclusions

The augmented Lagrangian algorithm
(method of multipliers)

- Requires only SOSC
  (equality constraints – noncriticality) for
  - Primal-dual Q-linear rate of convergence
  - Primal Q-linear rate of convergence
  - The rates are inversely proportional to c_k
- Not needed:
  - LICQ, Strict complementarity
- Reliably computes good MPCC solutions

Details:

SIAM Journal on Optimization, Vol. 22, 384–407
(Rates of convergence theory , with D. Fernández)

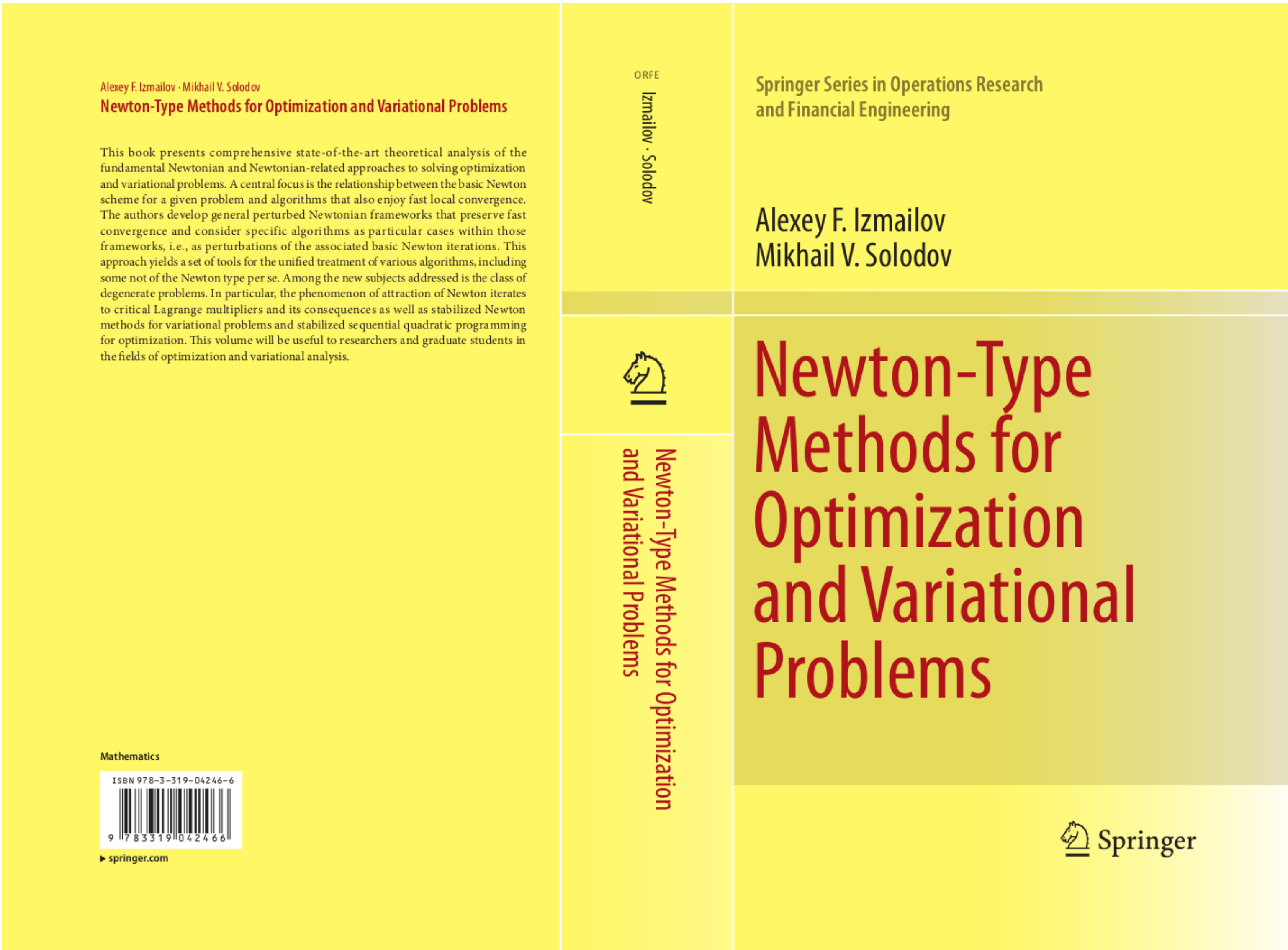
SIAM Journal on Optimization, Vol. 22, 1579–1606
(MPCC and numerics, with Uskov & Izmailov)

Computational Optimization and Applications, 2014
(Equality constraints, with Kurennoy & Izmailov)

or

`http://www.impa.br/~optim/solodov.html`

(600 pages approx.)



Alexey F. Izmailov · Mikhail V. Solodov

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Mathematics

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